Introduction to Model Theory

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Three lectures

Lecture 1 A short survey of model theory, with a proof of the zero-one law on finite models, using Ehrenfeucht-Fraisse games.

Lecture 2 Applications of Hintikka sets, with a proof of the Compactness Theorem and of the Interpolation Theorem.

Lecture 3 Model theory of databases? Dependence logic. Compactness and Completeness results, using model theoretic methods.

Classical results that set the stage in model theory

Theorem (Tarski 1930)

There is a complete axiomatization of $(\mathbb{R}, +, \cdot)$ *and of* $(\mathbb{C}, +, \cdot)$.

Theorem (Gödel 1931)

There is no complete axiomatization of $(\mathbb{N}, +, \cdot)$.

Theorem (Gödel 1930)

A sentence ϕ *has a proof from the axioms of a theory* T *if and only if every model of* T *is a model of* ϕ.

Basic results of model theory

Theorem

1. *Compactness Theorem: If T is a set of sentences and every finite subset of T has a model, then T has a model.*
2. *Upward Löwenheim-Skolem Theorem: If T is a set of sentences and T has an infinite model, then T has models of all infinite cardinalities $\geq |T|$.*
3. *Downward Löwenheim-Skolem Theorem: If T is a countable set of sentences and T has an infinite model, then T has a countable model.*

First order theories

- Examples: Dense linear order, successor function, algebraically closed fields.
- Complete theories: Decide every sentence in the signature of the theory.
- $\aleph_0$-categorical theories: All countably infinite models are isomorphic. Always complete (assuming the theory has only infinite models).
- Decidable theories. Complete (in particular, the $\aleph_0$-categorical) theories are decidable.

Zero-one law

- Probabilities of sentences in finite models

$$P(\phi) = \lim_{n \to \infty} \frac{|\text{structures that satisfy } \phi \text{ on } n|}{|\text{all structures on } n|}$$

  Zero-one law: Always $P(\phi) = 0$ or $P(\phi) = 1$.

- Extension axioms in graphs, random graphs: $\forall x \forall y (y \neq x \rightarrow \exists z (zEx \wedge \neg zEy))$ etc.
- $\aleph_0$-categoricity of the theory of random graphs: Back-and-forth.
- Zero-one law as a consequence of $\aleph_0$-categoricity: The extension axioms have probability 1 and decide every sentence. Hence every sentence has limit probability 0 or 1.

Example

Suppose **P** says "there is an isolated vertex". Then $\mu(\mathbf{P}) = 0$, as the following calculation shows. The isolated vertex can be chosen in n ways and the remaining vertices can form a graph in any way.

$$\mu_n(\mathbf{P}) \leq \frac{n \cdot 2^{\binom{n-1}{2}}}{2^{\binom{n}{2}}} = \frac{n}{2^{n-1}} \to 0.$$

□

Analysis of the structure of models

- Unions of chains of models: Union of an elementary chain is an elementary extension of each model in the chain.
- Type: Property of an “ideal" element.
- Satisfying a type in a model, or perhaps in an elementary extension.
- Omitting a type in a model. Omitting Types Theorem.
- Saturated models. Existence of them. Uniqueness.

Morley's Theorem

- κ-categoricity: All models of cardinality κ are isomorphic. (E.g. successor function when κ uncountable).
- Morley's Theorem: A theory is κ-categorical for some uncountable κ if and only if it is κ-categorical for all uncountable κ

Elements of the proof of Morley's Theorem

- Stability of complete first order theories: Not too many types.
- $\aleph_1$-categorical theories are stable: Suppose not stable in ω, e.g. has a model of cardinality $\aleph_1$ with countably many constants and $\aleph_1$ types satisfied. By an indiscernible set argument (based on Ramsey's Theorem) the theory has also a model of size $\aleph_1$ satisfying only countably many types. This contradicts $\aleph_1$-categoricity.
- Suppose the theory is $\aleph_1$-categorical, hence stable in ω. Can construct a saturated model of size $\aleph_1$. Hence all models of size $\aleph_1$ are saturated. Can show all models of uncountable cardinality are saturated. Hence κ-categorical for all $\kappa > \omega$.

Shelah's stability theory I

- Shelah's classification theory. Theories with good structure theory vs. theories with “non-structure" i.e. no reasonable structure theory.
- Geometric model theory: Algebraic closure on a strongly minimal set.
- Shelah's Main Gap: Complete theories are divided into two classes: those with good structure and those without any reasonable structure, and there are no others.

Shelah's stability theory II

- The structure case: successor function, infinitely many disjoint infinite unary predicates, algebraically closed fields, vector spaces over a fixed finite field.
- The non-structure case: number theory, set theory
- Hodges (p. 273): "On any reckoning this [Shelah's Main Gap] is one of the major achievements in mathematical logic since Aristotle".