Introduction to Model Theory

Jouko Väänänen¹,²

¹Department of Mathematics and Statistics, University of Helsinki

²Institute for Logic, Language and Computation, University of Amsterdam

Beijing, June 2016
Three lectures

Lecture 1  A short survey of model theory, with a proof of the zero-one law on finite models, using Ehrenfeucht-Fraisse games.

Lecture 2  Applications of Hintikka sets, with a proof of the Compactness Theorem and of the Interpolation Theorem.

Classical results that set the stage in model theory

Theorem (Tarski 1930)
There is a complete axiomatization of \((\mathbb{R}, +, \cdot)\) and of \((\mathbb{C}, +, \cdot)\).

Theorem (Gödel 1931)
There is no complete axiomatization of \((\mathbb{N}, +, \cdot)\).

Theorem (Gödel 1930)
A sentence \(\phi\) has a proof from the axioms of a theory \(T\) if and only if every model of \(T\) is a model of \(\phi\).
Basic results of model theory

**Theorem**

1. **Compactness Theorem**: If $T$ is a set of sentences and every finite subset of $T$ has a model, then $T$ has a model.

2. **Upward Löwenheim-Skolem Theorem**: If $T$ is a set of sentences and $T$ has an infinite model, then $T$ has models of all infinite cardinalities $\geq |T|$.

3. **Downward Löwenheim-Skolem Theorem**: If $T$ is a countable set of sentences and $T$ has an infinite model, then $T$ has a countable model.
First order theories

- **Examples**: Dense linear order, successor function, algebraically closed fields.
- **Complete theories**: Decide every sentence in the signature of the theory.
- **$\aleph_0$-categorical theories**: All countably infinite models are isomorphic. Always complete (assuming the theory has only infinite models).
- **Decidable theories**: Complete (in particular, the $\aleph_0$-categorical) theories are decidable.
Zero-one law

Proportions of sentences in finite models

\[ P(\phi) = \lim_{n \to \infty} \frac{|\text{structures that satisfy } \phi \text{ on } n|}{|\text{all structures on } n|} \]

Zero-one law: Always \( P(\phi) = 0 \) or \( P(\phi) = 1 \).

Extension axioms in graphs, random graphs:
\[ \forall x \forall y (y \neq x \rightarrow \exists z (zEx \land \neg zEy)) \text{ etc.} \]

\( \aleph_0 \)-categoricity of the theory of random graphs: Back-and-forth.

Zero-one law as a consequence of \( \aleph_0 \)-categoricity: The extension axioms have probability 1 and decide every sentence. Hence every sentence has limit probability 0 or 1.
Example

Suppose $P$ says "there is an isolated vertex". Then $\mu(P) = 0$, as the following calculation shows. The isolated vertex can be chosen in $n$ ways and the remaining vertices can form a graph in any way.

$$\mu_n(P) \leq \frac{n \cdot 2^{\binom{n-1}{2}}}{2^{\binom{n}{2}}} = \frac{n}{2^{n-1}} \to 0.$$
Analysis of the structure of models

- **Unions of chains** of models: Union of an elementary chain is an elementary extension of each model in the chain.
- **Type**: Property of an “ideal” element.
- **Satisfying a type** in a model, or perhaps in an elementary extension.
- **Omitting a type** in a model. Omitting Types Theorem.
- **Saturated models**. Existence of them. Uniqueness.
Morley’s Theorem

- $\kappa$-categoricity: All models of cardinality $\kappa$ are isomorphic. (E.g. successor function when $\kappa$ uncountable).

- Morley’s Theorem: A theory is $\kappa$-categorical for some uncountable $\kappa$ if and only if it is $\kappa$-categorical for all uncountable $\kappa$
Elements of the proof of Morley’s Theorem

- **Stability** of complete first order theories: Not too many types.

- **$\aleph_1$-categorical theories are stable:** Suppose not stable in $\omega$, e.g. has a model of cardinality $\aleph_1$ with countably many constants and $\aleph_1$ types satisfied. By an indiscernible set argument (based on Ramsey’s Theorem) the theory has also a model of size $\aleph_1$ satisfying only countably many types. This contradicts $\aleph_1$-categoricity.

- Suppose the theory is $\aleph_1$-categorical, hence stable in $\omega$. Can construct a saturated model of size $\aleph_1$. Hence all models of size $\aleph_1$ are saturated. Can show all models of uncountable cardinality are saturated. Hence $\kappa$-categorical for all $\kappa > \omega$. 
Shelah’s classification theory. Theories with good structure theory vs. theories with “non-structure" i.e. no reasonable structure theory.

Geometric model theory: Algebraic closure on a strongly minimal set.

Shelah’s Main Gap: Complete theories are divided into two classes: those with good structure and those without any reasonable structure, and there are no others.
Shelah’s stability theory II

The **structure** case: successor function, infinitely many disjoint infinite unary predicates, algebraically closed fields, vector spaces over a fixed finite field.

The **non-structure** case: number theory, set theory

Hodges (p. 273): “On any reckoning this [Shelah’s Main Gap] is one of the major achievements in mathematical logic since Aristotle".