Hintikka sets and their applications

Jouko Väänänen[1,2]

[1]Department of Mathematics and Statistics, University of Helsinki

[2]Institute for Logic, Language and Computation, University of Amsterdam

Beijing, June 2016

We learn a model construction method.

A preliminary convention: Pushing negation inside

$$
\begin{array}{lcl}
\phi\neg & = & \neg\phi \text{ if } \phi \text{ atomic} \\
(\neg\phi)\neg & = & \phi \\
(\phi \wedge \psi)\neg & = & \neg\phi \vee \neg\psi \\
(\phi \vee \psi)\neg & = & \neg\phi \wedge \neg\psi \\
(\forall x\phi)\neg & = & \exists x\neg\phi \\
(\exists x\phi)\neg & = & \forall x\neg\phi
\end{array}
$$

Definition

L countable, C a countable set of new constant symbols and $L' = L \cup C$, $C = C$. A *Hintikka set* is any set H of L'-sentences, which satisfies:

1. $t = t \in H$ for every constant L'-term t. (Add $\neg c = c'$ for *strict Hintikka set* whenever $\mathfrak{s}(c) \neq \mathfrak{s}(c')$.)
2. If $\phi(t) \in H$, $\phi(t)$ atomic, and $t = t' \in H$, then $\phi(t') \in H$.
3. If $\neg\phi \in H$, then $\phi\neg \in H$.
4. If $\phi \vee \psi \in H$, then $\phi \in H$ or $\phi \in H$.
5. If $\phi \wedge \psi \in H$, then $\phi \in H$ and $\phi \in H$.
6. If $\exists x\phi(x) \in H$, then $\phi(c) \in H$ for some $c \in C$
7. If $\forall x\phi(x) \in H$, then $\phi(c) \in H$ for all $c \in C$
8. For every constant L'-term t there is $c \in C$ such that $t = c \in H$.
9. There is no atomic sentence ϕ such that $\phi \in H$ and $\neg\phi \in H$.

The Hintikka set H is a Hintikka set *for* a sentence ϕ if $\phi \in H$.

1. $t = t \in H$ for every constant L'-term t.
2. If $\phi(t) \in H$, $\phi(t)$ atomic, and $t = t' \in H$, then $\phi(t') \in H$.
3. If $\neg\phi \in H$, then $\phi\neg \in H$.
4. If $\phi \wedge \psi \in H$, then $\phi \in H$ and $\psi \in H$.
5. If $\phi \vee \psi \in H$, then $\phi \in H$ or $\psi \in H$.
6. If $\exists x\phi(x) \in H$, then $\phi(c) \in H$ for some $c \in C$
7. If $\forall x\phi(x) \in H$, then $\phi(c) \in H$ for all $c \in C$
8. For every constant L'-term t there is $c \in C$ such that $t = c \in H$.
9. There is no atomic ϕ such that $\phi \in H$ and $\neg\phi \in H$.

A basic lemma

Lemma

If there is a Hintikka set for ϕ, then ϕ has a model.

Note: Conversely, if ϕ has a model $\mathcal{M}$, then there is a Hintikka set for ϕ, built directly from formulas true in $\mathcal{M}$.

Note: A Hintikka set need not be complete. There may be ϕ such that neither $\phi \in H$ nor $\neg\phi \in H$.

Proof.

- $M = \{[c] : c \in C\}$, where $c \sim c'$ is defined as $c = c' \in H$.
- $c^{\mathcal{M}} = [c]$.
- Let $f^{\mathcal{M}}([c_{i_1}], \ldots, [c_{i_n}]) = [c]$ for $c \in C$ such that $f(c_{i_1}, \ldots, c_{i_n}) = c \in H$.
- For any constant term t there is a $c \in C$ such that $t = c \in H$. It is easy to see that $t^{\mathcal{M}} = [c]$.
- We let $\mathcal{M} \models R(t_1, \ldots, t_n)$ if and only if $R(t_1, \ldots, t_n) \in H$.
- By induction on $\phi(x_1, \ldots, x_n)$: if $d_1 \ldots, d_n \in C$ then:
  1. If $\phi(d_1, \ldots, d_n) \in H$, then $\mathcal{M} \models \phi(d_1, \ldots, d_n)$.
  2. If $\neg\phi(d_1, \ldots, d_n) \in H$, then $\mathcal{M} \not\models \phi(d_1, \ldots, d_n)$.

  In particular, $\mathcal{M} \models \phi$ for the ϕ we started with, since $\phi \in H$.

□

Whence Hintikka sets?

- How to find useful Hintikka sets?
- The tool is: consistency property.
- A consistency property is a set of (usually) finite sets S which are consistent and the consistency property has information about how to extend S to a Hintikka set, which will then give a model for S.

Definition

Let L be a countable signature, C a countable set of new constant symbols and $L' = L \cup C$. A *consistency property* is any set Δ of countable sets S of L-formulas, which satisfies the conditions:

1. If $S \in \Delta$, then $S \cup \{t = t\} \in \Delta$ for every constant L'-term t. For *strict Consistency Property* demand $S \cup \{\neg t = t'\} \in \Delta$ if $\mathfrak{s}(t) \neq \mathfrak{s}(t')$.
2. If $\phi(t) \in S \in \Delta$, $\phi(t)$ atomic, and $t = t' \in S$, then $S \cup \{\phi(t')\} \in \Delta$.
3. If $\neg\phi \in S \in \Delta$, then $S \cup \{\phi\neg\} \in \Delta$.
4. If $\phi \vee \psi \in S \in \Delta$, then $S \cup \{\phi\} \in \Delta$ or $S \cup \{\psi\} \in \Delta$
5. If $\phi \wedge \psi \in S \in \Delta$, then $S \cup \{\phi\} \in \Delta$ and $S \cup \{\psi\} \in \Delta$
6. If $\exists x\phi(x) \in S \in \Delta$, then $S \cup \{\phi(c)\} \in \Delta$ for some $c \in C$.
7. If $\forall x\phi(x) \in S \in \Delta$, then $S \cup \{\phi(c)\} \in \Delta$ for all $c \in C$.
8. For every constant L'-term t there is $c \in C$ such that $S \cup \{t = c\} \in \Delta$.
9. There is no atomic formula ϕ such that $\phi \in S$ and $\neg\phi \in S$.

The consistency property Δ is a consistency property *for* a set T of L-sentences if for all $S \in \Delta$ and all $\phi \in T$ we have $S \cup \{\phi\} \in \Delta$.

Existence of Hintikka sets

Lemma

Let T be a countable set of L-sentences. Suppose Δ is a consistency property for T. Then for any $S \in \Delta$ there is a Hintikka set H for T such that $S \subseteq H$.

Proof:

- Let *Trm* the set of all constant L'-terms.
- Let

$$\begin{array}{rcl} T & = & \{\phi_n : n \in \mathbb{N}\} \\ C & = & \{c_n : n \in \mathbb{N}\} \\ Trm & = & \{t_n : n \in \mathbb{N}\}. \end{array}$$

- Let $\{\psi_n : n \in \mathbb{N}\}$ be a list of all L'-formulas.
- We define H as the union of an increasing sequence $S_0, S_1, \ldots$, where $S_0 = S$.
- $S_{n+1} = S_n$, but:

1. If $n = 3^i$, then S_{n+1} is $S_n \cup \{\phi_i\} \in \Delta$.
2. If $n = 2 \cdot 3^i$, then S_{n+1} is $S_n \cup \{t_i = t_i\} \in \Delta$.
3. If $n = 4 \cdot 3^i \cdot 5^j \cdot 7^k$, $\psi_i = (t = t') \in S_n$, and $\psi_j = \phi(t) \in S_n$ with $\phi(t)$ atomic, then S_{n+1} is $S_n \cup \{\phi(t')\} \in \Delta$.
4. If $n = 8 \cdot 3^i \cdot 5^k$ and $\psi_i = \neg\psi \in S_n$, then S_{n+1} is $S_n \cup \{\psi\neg\}$.
5. If $n = 16 \cdot 3^i \cdot 5^k$ and $\psi_i = \psi \vee \phi \in S_n$, then S_{n+1} is $S_n \cup \{\psi\}$ or $S_n \cup \{\phi\}$, whichever is in Δ.
6. If $n = 32 \cdot 3^i \cdot 5^j \cdot 7^k$, $j \in \{0, 1\}$ and $\psi_i = \phi_0 \wedge \phi_1 \in S_n$, then S_{n+1} is $S_n \cup \{\phi_j\}(\in \Delta)$.
7. If $n = 64 \cdot 3^i \cdot 7^k$ and $\psi_i = \exists x\phi \in S_n$, then S_{n+1} is $S_n \cup \{\phi(c)\}$ for such $c \in C$ that $S_{n+1} \in \Delta$.
8. If $n = 128 \cdot 3^i \cdot 5^j \cdot 7^k$ and $\psi_i = \forall x\phi \in S_n$, then S_{n+1} is $S_n \cup \{\phi(c_j)\}(\in \Delta)$.
9. If $n = 256 \cdot 3^i$, then S_{n+1} is $S_n \cup \{t_i = c\}$ for such $c \in C$ that $S_{n+1} \in \Delta$.

Clearly $\bigcup_n S_n$ is a Hintikka set for T.

Consistency property from “consistency"

Lemma

The set Δ of finite sets S of sentences such that $S \nvdash \perp$ is a consistency property.

Proof:

1. Clearly, if $S \in \Delta$, then $S \cup \{t = t\} \in \Delta$ for every constant L'-term t.
2. Clearly, if $\phi(t) \in S \in \Delta$, $\phi(t)$ atomic, and $t = t' \in S$, then $S \cup \{\phi(t')\} \in \Delta$.
3. Suppose $\neg\phi \in S \in \Delta$, but $S \cup \{\phi\neg\} \vdash \perp$. Then $S \vdash \perp$, contradiction.

- Suppose $\phi \vee \psi \in S \in \Delta$ but $S \cup \{\phi\} \vdash \bot$ and $S \cup \{\psi\} \vdash \bot$. Then $S \vdash \bot$, contradiction.
- Suppose $\phi \wedge \psi \in S \in \Delta$ but $S \cup \{\phi\} \vdash \bot$ or $S \cup \{\psi\} \vdash \bot$. Then $S \vdash \bot$, contradiction.
- Suppose $\exists x \phi(x) \in S \in \Delta$ but $S \cup \{\phi(c)\} \vdash \bot$ for all $c \in C$. Then $S \vdash \bot$, because we can choose c so that it does not occur in S. Contradiction.
- Suppose $\forall x \phi(x) \in S \in \Delta$ but $S \cup \{\phi(c)\} \vdash \bot$ for some $c \in C$. Then $S \vdash \bot$, contradiction.
- Let us consider a constant L'-term t. There is $c \in C$ such that $S \cup \{t = c\} \in \Delta$.
- There is no atomic formula ϕ such that $\phi \in S$ and $\neg\phi \in S$, because $\{\phi, \neg\phi\} \vdash \bot$.

The Completeness Theorem

Theorem

TFAE for all ϕ:

1. $\models \phi$ *i.e.* ϕ *is true in all models.*
2. $\vdash \phi$ *i.e.* ϕ *has a proof.*

Proof.

If ϕ has a proof, then clearly $\models \phi$. If ϕ does not have a proof, then $\neg\phi \nvdash \bot$. So $\{\neg\phi\} \in \Delta$ for the Δ in the previous lemma. Hence $\neg\phi$ has a model and $\not\models \phi$. □

Consistency property from "finitely consistent"

Lemma

The set Δ of sets S of sentences such that only finitely many constants in C occur in S and every finite subset of S has a model, is a consistency property.

The Compactness Theorem

Theorem

If T is a countable set of sentences such that every finite subset of T has a model then T itself has a model.

Proof.

By assumption $S \in \Delta$ for the Δ in the previous lemma. Hence S has a model. $\square$

Interpolation

Craig Interpolation Theorem

Theorem

We assume that L_1 and L_2 are vocabularies. Suppose $\models \phi \to \psi$, where ϕ is an L_1-sentence and ψ is an L_2-sentence. Then there is an $L_1 \cap L_2$-sentence θ ("interpolant") such that

1. $\models \phi \to \theta$
2. $\models \theta \to \psi$

Proposition

Suppose ϕ depends only on R in the sense that

$$\mathcal{M} \models \phi \iff \mathcal{N} \models \phi$$

whenever $\mathcal{M}$ and $\mathcal{N}$ have the same domain and the same interpretation of R. Then $\models \phi \leftrightarrow \psi$ where no non-logical symbols except R occurs in ψ.

Proof.

Let $S_1, ..., S_n$ be the non-logical symbols in ϕ or ψ in addition to R. Let ϕ' be the result of replacing each S_i in ϕ by a new symbol S_i'. Now

$$\models \phi \to \phi'.$$

By the Interpolation Theorem there is θ containing only R such that $\models \phi \to \theta$ and $\models \theta \to \phi'$. Hence $\models \phi \leftrightarrow \theta$. □

Theorem (Beth Definability Theorem)

Suppose the predicate S depends only on R in the sense that

$$\phi(R,S)\wedge\phi(R,S')\models\forall x(S(x)\leftrightarrow S'(x)).$$

Then there is $\theta(R,x)$ *where S does not occur such that* $\phi(R,S)\models\forall x(S(x)\leftrightarrow\theta(R,x))$.

Proof.

By assumption

$$\models(\phi(R,S)\wedge S(c))\rightarrow(\phi(R,S')\rightarrow S'(c)).$$

Let $\theta(R,c)$ be such that

$$\models(\phi(R,S)\wedge S(c))\rightarrow\theta(R,c)\text{ and }\models\theta(R,c)\rightarrow(\phi(R,S')\rightarrow S'(c)).$$

Then $\phi(R,S)\models\forall x(S(x)\leftrightarrow\theta(R,x))$. □

Example

Interpolation fails in finite models. Suppose ϕ is as follows:

$$\begin{array}{l} \forall x \exists y S(x,y) \wedge \\ \forall x \forall y \forall y' ((S(x,y) \wedge S(x,y')) \rightarrow y = y') \\ \forall x \forall y (S(x,y) \rightarrow (\neg x = y \wedge S(y,x))) \end{array}$$

Suppose ψ is

$$\begin{array}{l} \exists z [S'(z,z) \wedge \forall x \exists y S'(x,y) \wedge \\ \forall x \forall y \forall y' ((S'(x,y) \wedge S'(x,y')) \rightarrow y = y') \\ \forall x \forall y ((S'(x,y) \wedge \neg x = z) \rightarrow (\neg x = y \wedge S'(y,x)))] \end{array}$$

Then $\models \phi \rightarrow \neg\psi$ but if θ is an interpolant, then θ is an identity-sentence which is true in exactly the finite models with even cardinality, which is impossible.

Now: Proof of the Interpolation Theorem.

Craig Interpolation Theorem

Theorem

We assume that L_1 and L_2 are vocabularies. Suppose $\models \phi \to \psi$, where ϕ is an L_1-sentence and ψ is an L_2-sentence. Then there is an $L_1 \cap L_2$-sentence θ ("interpolant") such that

1. $\models \phi \to \theta$
2. $\models \theta \to \psi$

Proof of Interpolation

- Let us assume that the claim of the theorem is false and derive a contradiction. Since $\models \phi \to \psi$, the set $\{\phi, \neg\psi\}$ has no models. (We may assume that ϕ is consistent and that ψ is not valid.) We construct a consistency property for $\{\phi, \neg\psi\}$.
- Let $L = L_1 \cap L_2$. Suppose $C = \{c_n : n \in \mathbb{N}\}$ is a set of new constant symbols.
- Given a set S of sentences, let S_1 consists of all $L_1 \cup C$-sentences in S and let S_2 consists of all $L_2 \cup C$-sentences in S.

Separation

Definition

Let us say that θ separates S_1 and S_2 if

1. $S_1 \models \theta$,
2. $S_2 \models \neg\theta$,
3. θ is an L-sentence.

Our consistency property

Definition

Let Δ consist of all finite sets S of L-sentences such that $S = S_1 \cup S_2$ and:

($\star$) There is no $L \cup C$-sentence that separates S_1 and S_2.

Δ is a consistency property

Case 0. $\{\phi, \neg\psi\} \in \Delta$. True by assumption.

Case 1. Suppose $S \in \Delta$ and consider $c = c$, where, for example, $c \in L_1 \cup C$. We let $S_1' = S_1 \cup \{c = c\}$ and $S_2' = S_2 \cup \{c = c\}$. Suppose $\theta(c_0, \ldots, c_{m-1})$ separates S_1' and S_2'. Then clearly also $\theta(c_0, \ldots, c_{m-1})$ separates S_1 and S_2, a contradiction.
Case 2. Suppose $\phi(c) \in S \in \Delta$, $\phi(c)$ atomic, and $c = c' \in S$. Clearly $S \cup \{\phi(c')\} \in \Delta$.
Case 3. Negation: trivial.

Case 4. Consider $\phi \vee \psi$, such that, for example, $\phi \vee \psi \in S_1$. We claim that either the sets $S_1 \cup \{\phi\}$ and S_2 satisfy ($\star$), or the sets $S_1 \cup \{\phi\}$ and S_2 satisfy ($\star$). Otherwise there some θ that separates $S_1 \cup \{\phi\}$ and S_2, and some θ' that separates $S_1 \cup \{\psi\}$ and S_2. Let $\theta^* = \theta \vee \theta'$. Then θ^* separates S_1 and S_2 contrary to assumption.

Case 5. Consider $\phi \wedge \psi$ where for example $\phi \wedge \psi \in S_1$. Let $S_1' = S_1 \cup \{\phi\}$ and $S_2' = S_2$. If θ separates S_1' and S_2', then clearly θ also separates S_1 and S_2. Hence $S \cup \{\phi\} \in \Delta$. Similarly for ψ.

Case 6. Consider $S \in \Delta$ and $\exists x \phi(x) \in S_1$. Let $c_0 \in C$ be such that c does not occur in S. We claim that the sets $S_1 \cup \{\phi(c_0)\}$ and S_2 satisfy $(\star)$. Otherwise there is some $\theta(c_0, \ldots, c_{m-1})$ that separates $S_1 \cup \{\phi(c_0)\}$ and S_2. Let[1] $\theta'(c_1, \ldots, c_{m-1}) = \exists x \theta(x, c_1, \ldots, c_{m-1})$. We show that $\theta'(c_1, \ldots, c_{m-1})$ separates S_1 and S_2, a contradiction. Checking this:

- $S_1 \models \theta'(c_1, \ldots, c_{m-1})$:

$$S_1 \cup \{\phi(c_0)\} \models \theta(c_0, \ldots, c_{m-1}) \text{ by assumption}$$

$$S_1 \models \phi(c_0) \to \theta(c_0, \ldots, c_{m-1})$$

$$S_1 \models \forall x(\phi(x) \to \theta(x, c_1, \ldots, c_{m-1}))$$

$$S_1 \models \exists x \phi(x) \to \exists x \theta(x, c_1, \ldots, c_{m-1})$$

$$S_1 \models \exists x \theta(x, c_1, \ldots, c_{m-1}) \text{ as } S_1 \models \exists x \phi(x)$$

$$S_1 \models \theta'(c_1, \ldots, c_{m-1})$$

[1] If c_0 does not occur in θ, then we take $\theta' = \theta$.

Case 6. (Contd.)

- $S_2 \models \neg\theta'(c_0, \ldots, c_{m-1})$:

$$S_2 \models \neg\theta(c_0, \ldots, c_{m-1})$$

$$S_2 \models \forall x \neg\theta(x, c_1, \ldots, c_{m-1})$$

$$S_2 \models \neg\exists x \theta(x, c_1, \ldots, c_{m-1})$$

$$S_2 \models \neg\theta'(c_1, \ldots, c_{m-1})$$

Case 6. (Contd.) Consider $S \in \Delta$ and $\exists x\phi(x) \in S_2$. Let $c_0 \in C$ be such that c_0 does not occur in S. We claim that the sets S_1 and $S_2 \cup \{\phi(c_0)\}$ satisfy $(\star)$. Otherwise there is some $\theta(c_0, \ldots, c_{m-1})$ that separates S_1 and $S_2 \cup \{\phi(c_0)\}$. Let[2] $\theta'(c_1, \ldots, c_{m-1}) = \forall x\theta(x, c_1, \ldots, c_{m-1})$. We show that $\theta'(c_1, \ldots, c_{m-1})$ separates S_1 and S_2, a contradiction. Checking this:

- $S_1 \models \theta'(c_1, \ldots, c_{m-1})$:

$$S_1 \models \theta(c_0, \ldots, c_{m-1})$$

$$S_1 \models \forall x\theta(x, c_1, \ldots, c_{m-1})$$

$$S_1 \models \theta'(c_1, \ldots, c_{m-1})$$

[2] If c_0 does not occur in θ, we choose $\theta' = \theta$.

- $S_2 \models \neg\theta'(c_1, \ldots, c_{m-1})$:

$$S_2 \cup \{\phi(c_0)\} \models \neg\theta(c_0, \ldots, c_{m-1})$$
$$S_2 \models \phi(c_0) \to \neg\theta(c_0, \ldots, c_{m-1})$$
$$S_2 \models \forall x(\phi(x) \to \neg\theta(x, c_1, \ldots, c_{m-1}))$$
$$S_2 \models \exists x\phi(x) \to \exists x\neg\theta(x, c_1, \ldots, c_{m-1})$$
$$S_2 \models \exists x\neg\theta(x, c_1, \ldots, c_{m-1})$$
$$S_2 \models \neg\theta'(c_1, \ldots, c_{m-1})$$

Case 7. Consider $S \in \Delta$, $\phi(c_0)$, where $c_0 \in C$ and $\forall x \phi(x) \in S_1$. Exercise!

Case 7. (Contd.) Consider $\phi(c_0)$, where $c_0 \in C$ and $\forall x \phi(x) \in S_2$. Exercise!

Case 8. We are given a term t. Let c be a constant not in S. Then $S \cup \{t = c\} \in \Delta$.

Case 9. Suppose $\phi \in S$ and $\neg\phi \in S$, where $S \in \Delta$.

Case 1: $\phi \in S_1$ and $\neg\phi \in S_1$. Let θ be $\neg\forall x(x = x)$. Then θ separates S_1 and S_2, contradiction.

Case 2: $\phi \in S_2$ and $\neg\phi \in S_2$. Let θ be $\forall x(x = x)$. Then θ separates S_1 and S_2, contradiction.

$\square$