Topological semantics of modal logic

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Organization

- 5 lectures + tutorials/Q & A sessions.
- Exercise sheets will be provided each day.
- There will be a written exam at the end of the course.

Literature

- P. Blackburn, M. de Rijke, and Y. Venema, Modal Logic, Cambridge University Press 2001.
- J. van Benthem and G. Bezhanishvili, Modal Logic of Space, Handbook of Spatial Logics, 2007.

Prerequisites

- Basic knowledge of modal logic.
- Familiarity with some basic concepts of general topology.

Content

- Overview of relational semantics of modal logic,
- Topological semantics,
- Topo-bisimulations,
- Topo-canonical models,
- Basic completeness results for topological semantics,
- McKinsey-Tarski theorem,
- Derived set operator semantics,
- Topological semantics of modal fixed-point logic (time permitting).

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- In this course we will concentrate on spatial/topological modal logic.

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One of the early reference along McKinsey and Tarski is Tang Tsao Chen (1938).

Early references

- T. Tsao-Chen, Algebraic postulates and a geometric interpretation for the Lewis calculus of strict implication, *Bulletin of the American Mathematical Society*, vol. 44 (1938), pp. 737-744.

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- J. C. C. McKinsey and A. Tarski, The algebra of topology, *Annals of Mathematics*, vol. 45 (1944), pp. 141-191.

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Part 1: Modal logic and its relational semantics

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We assume the standard abbreviations: $\varphi \wedge \psi := \neg(\neg\varphi \vee \neg\psi)$, $\varphi \to \psi := \neg\varphi \vee \psi$, $\varphi \leftrightarrow \psi = (\varphi \to \psi) \wedge (\psi \to \varphi)$.

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Most importantly: $\Box\varphi := \neg\Diamond\neg\psi$.

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Let $\mathfrak{M} = (W, R, V)$ be a Kripke model and $w \in W$. We defined by induction when a formula φ is satisfied at w in $\mathfrak{M}$, written $\mathfrak{M}, w \models \varphi$:

$$\mathfrak{M}, w \models \bot \text{ never}$$

$$\mathfrak{M}, w \models p \text{ iff } w \in V(p),$$

$$\mathfrak{M}, w \models \varphi \vee \psi \text{ iff } \mathfrak{M}, w \models \varphi \text{ or } \mathfrak{M}, w \models \psi,$$

$$\mathfrak{M}, w \models \neg\varphi \text{ iff } \mathfrak{M}, w \not\models \varphi,$$

$$\mathfrak{M}, w \models \Diamond\varphi \text{ iff } \exists v \in W \text{ such that } wRv \text{ and } \mathfrak{M}, v \models \varphi.$$

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Löb defines transitive coversely well-founded frames and Grz defines transitive Nötherian frames.

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Proposition. Let (W', R') be a generated subframe of (W, R). Then for each modal formula φ we have

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Disjoint unions

Let $\mathcal{A} = \{(W_i, R_i) : i \in I\}$ be a family of Kripke frames. The disjoint union $\biguplus_{i \in I}(W_i, R_i)$ of $\mathcal{A}$ is the frame (W, R) such that $W = \biguplus_{i \in I} W_i$ and $R = \bigcup_{i \in I} R_i$.

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Theorem. Bisimilar points satisfy the same modal formulas.

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Theorem (Hennessy-Milner) On image finite models if two points satisfy the same modal formulas, then they are bisimilar.

Theorem (van Benthem) On (finite) Kripke models modal logic is a bisimulation invariant-fragment of First-Order logic.

Normal modal logics

A normal modal logic is a set of formulas that contains the K-axioms

$$\Box(p \to q) \to (\Box p \to \Box q)$$

and is closed under the rules of Modus Ponens (MP), Necessitation (N) and Uniform Substitution (US)

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The least normal modal logic is called the basic modal logic and is denoted by K.

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- $\mathsf{KT} = \mathsf{K} + (p \to \Diamond p)$,
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We often write $\vdash_L \varphi$ to mean $\varphi \in L$.

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- K4 is sound and complete with respect to the class of transitive frames.
- S4 is sound and complete with respect to the class of reflexive and transitive frames.

Sahlqvist Theorem. Every logic axiomatized by Sahlqvist formulas is Kripke complete wrt first-order definable frames.

The finite model property

A modal logic L is said to have the finite model property (FMP) if there is a class $\mathcal{C}$ of finite frames such that for every formula φ we have:

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KT, K4 and S4 have the finite model property.

Part 2: Topological semantics

Topological semantics

- Topological semantics of modal logic extends relational semantics.
- It provides a richer semantic landscape by bringing in a geometric/spatial interpretations.
- There are important Kripke incomplete logics (e.g., the provability logic **GLP**) which are topologically complete.
- Topological models provide (interesting) semantics for knowledge and belief.

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It is easy to check that $\mathrm{Cl}(A) = X \setminus \mathrm{Int}(X \setminus A)$.

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Theorem. The following are equivalent.

- (X, τ) is Alexandroff,
- τ is closed under arbitrary intersections,
- Every point $x \in X$ has a least open neighbourhood (the intersection of all its open neighbourhoods).

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The real line is an example of a non-Alexandroff topological space.

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The semantics for modal formulas is defined by the following inductive definition, where p is a propositional variable:

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Since $\Diamond\varphi = \neg\Box\neg\varphi$, we have $[\![\Diamond\varphi]\!] = \mathrm{Cl}[\![\varphi]\!]$.

Example: Spoon

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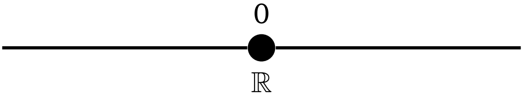
Validity of formulas in topological spaces is defined in the same way as for Kripke frames.

Topological semantics

Let $\nu(p) = (0, \infty)$.



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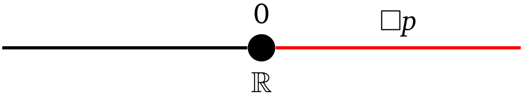

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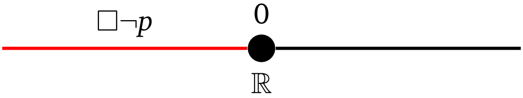

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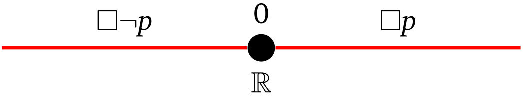

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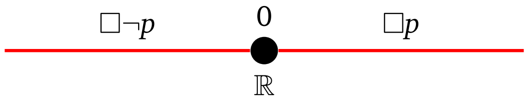

$$\nu(\Box p \vee \Box\neg p) \neq \mathbb{R}$$

Next lecture: Topological completeness

Theorem (McKinsey and Tarski, 1944)

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