Expressivity and Inference in Hybrid Logic

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Homework Sheet 1

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Exercise 1. Give the standard translation of $\Diamond\Diamond i \to \Diamond i$.

Model answer.

$$\begin{aligned}
& \mathrm{ST}_x(\Diamond\Diamond i \to \Diamond i) \\
=\ & \mathrm{ST}_x(\Diamond\Diamond i) \to \mathrm{ST}_x(\Diamond i) \\
=\ & \exists y(Rxy \wedge \mathrm{ST}_y(\Diamond i)) \to \mathrm{ST}_x(\Diamond i) \\
=\ & \exists y(Rxy \wedge \exists z(Ryz \wedge \mathrm{ST}_z(i))) \to \mathrm{ST}_x(\Diamond i) \\
=\ & \exists y(Rxy \wedge \exists z(Ryz \wedge i = z)) \to \mathrm{ST}_x(\Diamond i) \\
=\ & \exists y(Rxy \wedge \exists z(Ryz \wedge i = z)) \to \exists u(Rxu \wedge \mathrm{ST}_u(i)) \\
=\ & \exists y(Rxy \wedge \exists z(Ryz \wedge i = z)) \to \exists u(Rxu \wedge i = u)
\end{aligned}$$

Exercise 2. We say that a frame (W, R) is *convergent* (or *Church Rosser*) iff

$$\forall x \forall y \forall z(Rxy \wedge Rxz \to \exists w(Ryw \wedge Rzw)).$$

Show that modal formula $\Diamond\Box p \to \Box\Diamond p$ defines the class of convergent frames. That is, show (a) that this formula is valid on all convergent frames, and (b) that if a frame is *not* convergent, you can falsify this formula on it.

Model answer.

(a) Let (W, R) be an arbitrary convergent frame, let $\mathcal{M} = (W, R, V)$ be an arbitrary model on that frame, and let w be an arbitrary world in W.
Assume that $\mathcal{M}, w \vDash \Diamond\Box p$. Then there is some v such that wRv and $\mathcal{M}, v \vDash \Box p$.
Now consider an arbitrary world u such that wRu. Since (W, R) is convergent and we have that wRv and wRu, it follows that there is some world x such that vRx and uRx.
Moreover, since $\mathcal{M}, v \vDash \Box p$, from vRx it follows that $\mathcal{M}, x \vDash p$.
Since uRx holds as well, $\mathcal{M}, x \vDash p$ implies that $\mathcal{M}, u \vDash \Diamond p$, and since u was an arbitrary world such that wRu, it follows that $\mathcal{M}, w \vDash \Box\Diamond p$.
Consequently, $\mathcal{M}, w \vDash \Diamond\Box p \to \Box\Diamond p$.
Since $\mathcal{M}$ and w were arbitrarily chosen, it follows that $(W, R) \vDash \Diamond\Box p \to \Box\Diamond p$.

(b) Consider an arbitrary frame (W, R) that is *not* convergent: then there are worlds w, v, and u such that wRu and wRv, but there is no world z such that uRz and vRz.
We can define a valuation V on (W, R) such that the resulting model $\mathcal{M} = (W, R, V)$ falsifies $\Diamond\Box p \to \Box\Diamond p$: let $V(p)$ be $\{x \in W \mid vRx\}$ for some propositional letter p. That is p is true at all R-successors of v.
Then we have that $\mathcal{M}, v \vDash \Box p$ and hence $\mathcal{M}, w \vDash \Diamond\Box p$ (since wRv).
However, $\mathcal{M}, w \nvDash \Box\Diamond p$, because wRu and $\mathcal{M}, u \nvDash \Diamond p$, since p is only true at R-successors of v, and v and u have no R-successor in common.
Thus, we have that $\mathcal{M}, w \nvDash \Diamond\Box p \to \Box\Diamond p$.
Consequently, $(W, R) \nvDash \Diamond\Box p \to \Box\Diamond p$.

Exercise 3. We say that a frame (W, R) is *antisymmetric* iff

$$\forall x \forall y((Rxy \wedge Ryx) \rightarrow x = y).$$

Show that the pure hybrid formula $@_i\Box(\Diamond i \rightarrow i)$ defines the class of antisymmetric frames. That is, show (a) that this formula is valid on all antisymmetric frames, and (b) that if a frame is *not* antisymmetric, you can falsify this formula on it. (c) Can you think of another formula not containing @ that defines this class of frames?

Model answer.

(a) Let (W, R) be an arbitrary antisymmetric frame, let $\mathcal{M} = (W, R, V)$ be an arbitrary model on that frame, and let w be an arbitrary world in W.
Let v be the denotation of i under V. We then have that $\mathcal{M}, v \vDash i$.
Now consider an arbitrary world u such that vRu and assume that $\mathcal{M}, u \vDash \Diamond i$.
Since v is the denotation of i under V and hence the only world where i is true, $\mathcal{M}, u \vDash \Diamond i$ implies that uRv.
Because (W, R) is antisymmetric, from vRu and uRv it follows that $u = v$ and hence we have $\mathcal{M}, u \vDash i$ as well.
Consequently, $\mathcal{M}, u \vDash \Diamond i \rightarrow i$, and since u was an arbitrary world with vRu, it follows that $\mathcal{M}, v \vDash \Box(\Diamond i \rightarrow i)$.
Because v is the denotation of i under V, $\mathcal{M}, v \vDash \Box(\Diamond i \rightarrow i)$ implies $\mathcal{M}, w \vDash @_i\Box(\Diamond i \rightarrow i)$.
Since $\mathcal{M}$ and w were arbitrarily chosen, it follows that $(W, R) \vDash @_i\Box(\Diamond i \rightarrow i)$.

(b) Consider an arbitrary frame (W, R) that is *not* antisymmetric: then there are worlds w and v such that wRv and vRw but $w \neq v$.
We can define a valuation V on (W, R) such that the resulting model $\mathcal{M} = (W, R, V)$ falsifies $@_i\Box(\Diamond i \rightarrow i)$: let $V(i) = \{w\}$.
We have that $\mathcal{M}, v \vDash \Diamond i$ (since vRw and $\mathcal{M}, w \vDash i$).
However, $\mathcal{M}, v \nvDash i$, and hence $\mathcal{M}, v \nvDash \Diamond i \rightarrow i$.
It follows that $\mathcal{M}, w \nvDash \Box(\Diamond i \rightarrow i)$ (since wRv and $\mathcal{M}, v \nvDash \Diamond i \rightarrow i$) and hence we have that $\mathcal{M}, w \nvDash @_i\Box(\Diamond i \rightarrow i)$ (since w is the denotation of i under V).
Consequently, $(W, R) \nvDash @_i\Box(\Diamond i \rightarrow i)$.

(c) $i \rightarrow \Box(\Diamond i \rightarrow i)$

Exercise 4. Let $\mathcal{M} = (W, R, V)$ and $\mathcal{M}' = (W', R', V')$ be models for the basic hybrid language (with just one $\Box$ and $\Diamond$), and let Z be a bisimulation-with-constants between $\mathcal{M}$ and $\mathcal{M}'$. Show that for all basic hybrid formulas φ, and all worlds w in $\mathcal{M}$ and w' in $\mathcal{M}'$ such that w is bisimilar to w' we have that:

$$\mathcal{M}, w \vDash \varphi \text{ iff } \mathcal{M}', w' \vDash \varphi.$$

Model answer.

The proof runs by induction on the structure of φ.

Base step: We need to show that for all proposition letters and nominals in PROP$\cup$NOM, and for all worlds w in W and w' in W' such that wZw':

$$\mathcal{M}, w \vDash a \text{ iff } \mathcal{M}', w' \vDash a.$$

Since we have that wZw', the claim is straightforward by the "atomic Harmony" clause of the definition of bisimulation-with-constants.

Induction hypothesis (IH): We assume that the claim holds for any proper subformula ψ of φ, and for all worlds w in W and w' in W' such that wZw':

$$\mathcal{M}, w \vDash \psi \text{ iff } \mathcal{M}', w' \vDash \psi.$$

Induction step:

($\neg$) Let $\varphi := \neg\psi$. We have: $\mathcal{M}, w \vDash \neg\psi \;\Leftrightarrow\; \mathcal{M}, w \nvDash \psi \;\overset{IH}{\Leftrightarrow}\; \mathcal{M}', w' \nvDash \psi \;\Leftrightarrow\; \mathcal{M}', w' \vDash \neg\psi$.

($\wedge$) Let $\varphi := \psi \wedge \theta$. We have: $\mathcal{M}, w \vDash \psi \wedge \theta \;\Leftrightarrow\; \mathcal{M}, w \vDash \psi$ and $\mathcal{M}, w \vDash \theta$
$\overset{IH}{\Leftrightarrow}\; \mathcal{M}', w' \vDash \psi$ and $\mathcal{M}', w' \vDash \theta \;\Leftrightarrow\; \mathcal{M}', w' \vDash \psi \wedge \theta$

($\vee$) Let $\varphi := \psi \vee \theta$. We have: $\mathcal{M}, w \vDash \psi \vee \theta \;\Leftrightarrow\; \mathcal{M}, w \vDash \psi$ or $\mathcal{M}, w \vDash \theta$
$\overset{IH}{\Leftrightarrow}\; \mathcal{M}', w' \vDash \psi$ or $\mathcal{M}', w' \vDash \theta \;\Leftrightarrow\; \mathcal{M}', w' \vDash \psi \vee \theta$.

($\rightarrow$) Let $\varphi := \psi \rightarrow \theta$. We have: $\mathcal{M}, w \vDash \psi \rightarrow \theta \;\Leftrightarrow\; \mathcal{M}, w \nvDash \psi$ or $\mathcal{M}, w \vDash \theta$
$\overset{IH}{\Leftrightarrow}\; \mathcal{M}', w' \nvDash \psi$ or $\mathcal{M}', w' \vDash \theta \;\Leftrightarrow\; \mathcal{M}', w' \vDash \psi \rightarrow \theta$.

($\Diamond$) Let $\varphi := \Diamond\psi$.
"$\Rightarrow$": $\mathcal{M}, w \vDash \Diamond\psi$
$\Rightarrow$ there exists some v such that wRv and $\mathcal{M}, v \vDash \psi$
$\Rightarrow$ by the "Forth" clause: there exists some v' such that $w'Rv'$ and vZv'
$\overset{IH}{\Rightarrow}$ there exists some v' such that $w'Rv'$ and $\mathcal{M}', v' \vDash \psi$
$\Rightarrow$ $\mathcal{M}', w' \vDash \Diamond\psi$

"$\Leftarrow$": $\mathcal{M}', w' \vDash \Diamond\psi$
$\Rightarrow$ there exists some v' such that $w'Rv'$ and $\mathcal{M}', v' \vDash \psi$
$\Rightarrow$ by the "Back" clause: there exists some v such that wRv and vZv'
$\overset{IH}{\Rightarrow}$ there exists some v such that wRv and $\mathcal{M}, v \vDash \psi$
$\Rightarrow$ $\mathcal{M}, w \vDash \Diamond\psi$

($\Box$) Let $\varphi := \Box\psi$. We have: $\mathcal{M}, w \vDash \Box\psi \;\Leftrightarrow\;$ for all v, if wRv then $\mathcal{M}, v \vDash \psi$
$\overset{IH}{\Leftrightarrow}$ for all v', if $w'Rv'$ then $\mathcal{M}', v' \vDash \psi \;\Leftrightarrow\; \mathcal{M}', w' \vDash \Box\psi$.

Note that the "Forth" and "Back" clauses of the definition of bisimulation-with-constants guarantee a one-to-one correspondence between the worlds v accessible from w and the worlds v' accessible from w, and they ensure that vZv' holds in each case.

($@_i$) Let $\varphi := @_i\psi$. We have: $\mathcal{M}, w \vDash @_i\psi \;\Leftrightarrow\; \mathcal{M}, v \vDash \psi$ where $V(i) = v$
$\overset{IH}{\Leftrightarrow}\; \mathcal{M}, v' \vDash \psi$ where $V'(i) = v' \;\Leftrightarrow\; \mathcal{M}', w' \vDash @_i\psi$

Note that by the "Nominal Constancy" clause of the definition of bisimulation-with-constants, we have vZv', since $V(i) = v$ and $V'(i) = v'$.