Expressivity and Inference in Hybrid Logic

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Homework Sheet 2

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Exercise 1. Give a tableau proof of $@_i i$.

Model answer.

1	$\neg @_j @_i i$	
2	$\neg @_i i$	$\neg @$ rule on 1
3	$@_i i$	equality rule
4	$\bot_{2,3}$	

Exercise 2. Give a tableau proof for the following derived rule.

$$\text{Modus Ponens} \qquad \frac{@_i\varphi \quad @_i(\varphi \to \psi)}{@_i\psi}$$

Model answer.

1		$@_i\varphi$	
2		$@_i(\varphi \to \psi)$	
3	$\neg @_i\varphi$		$@_i\psi$
4	$\bot_{1,3}$		

Exercise 3. Give a tableau proof of $(\Diamond p \wedge \Diamond\neg p) \to (\Box(q \to i) \to \Diamond\neg q))$. (You may make use of the derived Modus Ponens rule from Exercise 2.)

Model answer.

1		$\neg @_i((\Diamond p \wedge \Diamond\neg p) \to (\Box(q \to i) \to \Diamond\neg q))$		
2		$@_i(\Diamond p \wedge \Diamond\neg p)$		
2′		$\neg @_i\Box(q \to i) \to \Diamond\neg q)$		$\neg \to$ rule on 1
3		$@_i\Diamond p$		
3′		$@_i\Diamond\neg p$		$\wedge$ rule on 2
4		$@_i\Diamond j$		
4′		$@_j p$		$\Diamond$ rule on 3
5		$@_i\Diamond k$		
5′		$@_k\neg p$		$\Diamond$ rule on 3’
6		$@_i\Box(q \to i)$		
6′		$\neg @_i\Diamond\neg q$		$\neg \to$ rule on 2’
7		$@_j q$		$\neg\Diamond$ rule on 4 and 6’, then $\neg @$ rule
8		$@_j(q \to i)$		$\Box$ rule on 4 and 6
9	$\neg @_j q$		$@_j i$	$\to$ rule on 7 and 8
	$\bot_{7,9}$			

9	$\neg @_j q$	$@_j i$	$\rightarrow$ rule on 7 and 8
	$\bot_{7,9}$		
10		$@_k q$	$\neg\Diamond$ rule on 5 and 6', then $\neg @$ rule
11		$@_k(q \rightarrow i)$	$\Box$ rule on 5 and 6
12		$@_k i$	Modus Ponens on 10 and 11
13		$@_i p$	equality rule on 4' and 9
14		$@_i \neg p$	equality rule on 5' and 12
15		$\neg @_i p$	$\neg$ rule on 14
16		$\bot_{13,15}$	

Exercise 4. We say that a frame (W, R) is *convergent* (or *Church Rosser*) iff

$$\forall x \forall y \forall z (Rxy \wedge Rxz \rightarrow \exists w (Ryw \wedge Rzw)).$$

Show that the pure hybrid tense logical formula $(Fi \wedge Fj) \rightarrow @_j FPi$ defines the class of convergent frames. That is, show (a) that this formula is valid on all convergent frames, and (b) that if a frame is *not* convergent, you can falsify this formula on it.

Model answer.

(a) Let $(T, <)$ be an arbitrary convergent frame, let $\mathcal{M} = (T, <, V)$ be an arbitrary model on that frame, and let w be an arbitrary temporal instant in T. [Note that since we are working in temporal logic, we write T instead of W (to indicate that we are dealing with a set of temporal instants) and $<$ instead of R (to indicate that the accessibility relation is the earlier-later relation.)]

Assume that $\mathcal{M}, w \vDash Fi \wedge Fj$. Then we have that (i) $\mathcal{M}, w \vDash Fi$ and (ii) $\mathcal{M}, w \vDash Fj$.

(i) $\mathcal{M}, w \vDash Fi$ implies that there is some v such that $w < v$ and $\mathcal{M}, v \vDash i$.

(ii) $\mathcal{M}, w \vDash Fj$ implies that there is some u such that $w < u$ and $\mathcal{M}, u \vDash j$.

Since $(T, <)$ is convergent, from $w < v$ and $w < u$ it follows that there is some x such that $v < x$ and $u < x$.

Because $v < x$ and $\mathcal{M}, v \vDash i$, it follows that $\mathcal{M}, x \vDash Pi$; and because $u < x$, this implies that $\mathcal{M}, u \vDash FPi$.

Since $\mathcal{M}, u \vDash j$, the temporal instant u is the denotation of j under V, and hence from $\mathcal{M}, u \vDash FPi$ it follows that $\mathcal{M}, w \vDash @_j FPi$.

Consequently, $\mathcal{M}, w \vDash (Fi \wedge Fj) \rightarrow @_j FPi$.

Since $\mathcal{M}$ and w were arbitrarily chosen, it follows that $(T, <) \vDash (Fi \wedge Fj) \rightarrow @_j FPi$.

(b) Consider an arbitrary frame $(T, <)$ that is *not* convergent: then there are temporal instants w, v, and u such that $w < u$ and $w < v$, but there is no temporal instant z such that $u < z$ and $v < z$.

We can define a valuation V on $(T, <)$ such that the resulting model $\mathcal{M} = (T, <, V)$ falsifies $(Fi \wedge Fj) \rightarrow @_j FPi$: let $V(i) = u$ and $V(j) = v$.

We have $\mathcal{M}, w \vDash Fi$ (since $w < u$ and $\mathcal{M}, u \vDash i$) and likewise $\mathcal{M}, w \vDash Fj$ (since $w < v$ and $\mathcal{M}, v \vDash j$). So, $\mathcal{M}, w \vDash Fi \wedge Fj$.

However, $\mathcal{M}, v \nvDash FPi$, since there is no temporal instant that is later than both v and u.

It follows that $\mathcal{M}, w \nvDash @_j FPi$ (since v is the denotation of j under V).

Thus, we have that $\mathcal{M}, w \nvDash (Fi \wedge Fj) \rightarrow @_j FPi$.

Consequently, $(T, <) \nvDash (Fi \wedge Fj) \rightarrow @_j FPi$.

Exercise 5. (a) Give the standard translation of $\Box(\Box i \to i) \to \Box i$. (b) Explain why this formula does *not* define the same class of frames as $\Box(\Box p \to p) \to \Box p$, the Löb formula.

Model answer.

(a) $\mathrm{ST}_x(\Box(\Box i \to i) \to \Box i)$
$= \mathrm{ST}_x(\Box(\Box i \to i)) \to \mathrm{ST}_x(\Box i)$
$= \forall y(Rxy \to \mathrm{ST}_y(\Box i \to i)) \to \mathrm{ST}_x(\Box i)$
$= \forall y(Rxy \to (\mathrm{ST}_y(\Box i) \to \mathrm{ST}_y(i))) \to \mathrm{ST}_x(\Box i)$
$= \forall y(Rxy \to (\forall z(Ryz \to \mathrm{ST}_z(i)) \to \mathrm{ST}_y(i))) \to \mathrm{ST}_x(\Box i)$
$= \forall y(Rxy \to (\forall z(Ryz \to i = z) \to \mathrm{ST}_y(i))) \to \mathrm{ST}_x(\Box i)$
$= \forall y(Rxy \to (\forall z(Ryz \to i = z) \to i = y)) \to \mathrm{ST}_x(\Box i)$
$= \forall y(Rxy \to (\forall z(Ryz \to i = z) \to i = y)) \to \forall u(Rxu \to \mathrm{ST}_u(i))$
$= \forall y(Rxy \to (\forall z(Ryz \to i = z) \to i = y)) \to \forall u(Rxu \to i = u)$

(b) The hybrid formula $\Box(\Box i \to i) \to \Box i$ does not define the same class of frames that $\Box(\Box p \to p) \to \Box p$ defines. The Löb formula defines a *non-elementary* (non-first-order definable) class of frames. (See Example 3.9 in *Modal Logic*, by Blackburn, de Rijke, and Venema for a detailed discussion.) However, $\Box(\Box i \to i) \to \Box i$ is a pure formula, and *all* pure formulas define elementary (first-order definable) classes of frames.