Expressivity and Inference in Hybrid Logic

Patrick Blackburn

Frame Definability: Revision Notes

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1 Frame definability

A formula φ defines a class of frames F if the following holds:

$$(W, R) \models \varphi \quad \text{iff} \quad (W, R) \text{ belongs to } \mathsf{F}.$$

In words: a formula φ defines a class of frames F iff φ is valid on all the frames that belong to F, and invalid on all other frames.

2 Examples

Here are some examples comparing frame definability in ordinary modal logic, and in hybrid logic (using pure formulas).

Ordinary formulas	Frame class	First-order formula
$p \to \Diamond p$	reflexive	$\forall x Rxx$
Not definable	irreflexive	$\forall x \neg Rxx$
$\Diamond\Diamond p \to \Diamond p$	transitive	$\forall x \forall y \forall z((Rxy \wedge Ryz) \to Rxz)$
$\Diamond\Box p \to \Box\Diamond p$	convergent	$\forall x \forall y \forall z((Rxy \wedge Rxz) \to \exists w(Ryw \wedge Rzw))$
Not definable	trichotomous	$\forall x \forall y(Rxy \vee x = y \vee Ryx)$

Pure formulas	Frame class	First-order formula
$i \to \Diamond i$	reflexive	$\forall x Rxx$
$i \to \neg\Diamond i$	irreflexive	$\forall x \neg Rxx$
$\Diamond\Diamond i \to \Diamond i$	transitive	$\forall x \forall y \forall z((Rxy \wedge Ryz) \to Rxz)$
Not definable	convergent	$\forall x \forall y \forall z((Rxy \wedge Rxz) \to \exists w(Ryw \wedge Rzw))$
$@_i\Diamond j \vee @_i j \vee @_j\Diamond i$	trichotomous	$\forall x \forall y(Rxy \vee x = y \vee Ryx)$

3 Pure validity is first-order

All the pure formulas listed above define first-order conditions. This is not a coincidence. All pure formulas define first-order frame conditions. Here is the standard translation for pure formulas.

$$\begin{array}{lcl}
\mathrm{ST}_x(i) & = & i = x \\
\mathrm{ST}_x(\neg\varphi) & = & \neg\,\mathrm{ST}_x(\varphi) \\
\mathrm{ST}_x(\varphi \wedge \psi) & = & \mathrm{ST}_x(\varphi) \wedge \mathrm{ST}_x(\psi) \\
\mathrm{ST}_x(\varphi \vee \psi) & = & \mathrm{ST}_x(\varphi) \vee \mathrm{ST}_x(\psi) \\
\mathrm{ST}_x(\varphi \to \psi) & = & \mathrm{ST}_x(\varphi) \to \mathrm{ST}_x(\psi) \\
\mathrm{ST}_x(\langle R\rangle\varphi) & = & \exists y(Rxy \wedge \mathrm{ST}_y(\varphi)) \\
\mathrm{ST}_x([R]\varphi) & = & \forall y(Rxy \to \mathrm{ST}_y(\varphi)) \\
\mathrm{ST}_x(@_i\varphi) & = & \mathrm{ST}_i(\varphi)
\end{array}$$

The translation of a pure formula thus contains one free first-order variable x, and perhaps many first-order constants $i_1, \ldots, i_n$, one for each nominal $i_1, \ldots, i_n$. So saying that the pure formula is *valid* is the same as saying:

$$\forall x \forall i_1 \cdots \forall i_n \text{ST}_i(\varphi),$$

which is a first-order sentence. This says: "For all states x (that's the $\forall x$) and no matter which state all the constants name (that's the $\forall i_1 \cdots \forall i_n$) we have that $\text{ST}_i(\varphi)$ is true".

4 Ordinary modal validity is second-order

All the ordinary modal formulas listed above define first-order conditions. **This is a coincidence**. All ordinary modal formulas define second-order frame conditions. Here is the standard translation for ordinary modal formulas.

$$\begin{array}{lcl}
\text{ST}_x(p) & = & Px \\
\text{ST}_x(\neg\varphi) & = & \neg\text{ST}_x(\varphi) \\
\text{ST}_x(\varphi \wedge \psi) & = & \text{ST}_x(\varphi) \wedge \text{ST}_x(\psi) \\
\text{ST}_x(\varphi \vee \psi) & = & \text{ST}_x(\varphi) \vee \text{ST}_x(\psi) \\
\text{ST}_x(\varphi \to \psi) & = & \text{ST}_x(\varphi) \to \text{ST}_x(\psi) \\
\text{ST}_x(\langle R\rangle\varphi) & = & \exists y(Rxy \wedge \text{ST}_y(\varphi)) \\
\text{ST}_x([R]\varphi) & = & \forall y(Rxy \to \text{ST}_y(\varphi))
\end{array}$$

The translation of an ordinary modal formula φ thus contains one free first-order variable x, and many predicate constants $P_1, \ldots, P_n$, one for each propositional symbol $p_1, \ldots, p_n$ in φ. So saying that the ordinary modal formula is *valid* is the same as saying:

$$\forall x \forall P_1 \cdots \forall P_n \text{ST}_i(\varphi),$$

which is a second-order sentence. This says: "For all states x (that's the $\forall x$) and no matter which subsets all the predicates pick out (that's the $\forall P_1 \cdots \forall P_n$) we have that $\text{ST}_i(\varphi)$ is true".

5 A second-order example

So ordinary modal formulas define frames with *second-order* logic. It is true that in many familiar cases, these second-order formulas are equivalent to first-order formulas. However this is not always true. There are ordinary modal formulas that define genuinely second-order frame properties. A well known example is the Löb formula:

$$\Box(\Box p \to p) \to \Box p.$$

The Löb formula defines the class of frames in which $<$ is transitive and there are no infinitely increasing chains

$$t_1 < t_2 < t_3 < t_4 < t_5 < t_6 \ldots.$$

For a proof, see Example 3.9 of the Blue Book. You will also find a proof there (using the first-order Compactness Theorem) that this property is *not* definable in first-order logic. That is, "transitivity plus no-infinite-ascending-chains" is not an *elementary* (that is, first-order definable) property of frames.