Expressivity and Inference in Hybrid Logic

Patrick Blackburn

Modal Completeness Proof: Revision Notes

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These notes are to remind you what a completeness proof for ordinary modal logic, using the canonical model method, looks like. In Session 3 we will be looking at the completeness proof for hybrid logic, and as we shall see, there are a number of important differences.

1 Soundness and Completeness

Soundness: "Only valid formulas are provable"

$$A \text{ is provable } \Rightarrow A \text{ is valid}$$

Completeness: "All valid formulas are provable"

$$A \text{ is valid } \Rightarrow A \text{ is provable}$$

Weak soundness and completeness

Soundness: $\vdash \varphi \Rightarrow \models \varphi$.

Completeness: $\models \varphi \Rightarrow \vdash \varphi$

Strong soundness and completeness

Let Γ be a set of formulas. Then $\Gamma \vdash \varphi$ means that there is a finite set $\{\gamma_1, \ldots, \gamma_n\} \subseteq \Gamma$ such that $\vdash \gamma_1, \ldots, \gamma_n \rightarrow \varphi$.

$\Gamma \models \varphi$ means that for any model $\mathcal{M}$ and any state w, if $\mathcal{M}, w \models \Gamma$ then $\mathcal{M}, w \models \varphi$.
(Here $\mathcal{M}, w \models \Gamma$ means that all formulas in Γ are true at w in $\mathcal{M}$.)

Soundness: $\Gamma \vdash \varphi \;\Rightarrow\; \Gamma \models \varphi$.

Completeness: $\Gamma \models \varphi \;\Rightarrow\; \Gamma \vdash \varphi$

2 Soundness

Soundness: $\Gamma \vdash \varphi \;\Rightarrow\; \Gamma \Vdash \varphi$

We need to show that the axioms are valid, and that the rules preserve validity. (See Chapter 1.6 of *Modal Logic* by Blackburn, de Rijke, and Venema for more discussion.)

3 Completeness

Completeness: $\Gamma \Vdash \varphi \;\Rightarrow\; \Gamma \vdash \varphi$

Completeness is about model existence: "Every consistent set of sentences has a model."

Line of reasoning:
$\Gamma \nvdash \varphi \;\Rightarrow\; \Gamma \cup \{\neg\varphi\}$ is consistent $\;\Rightarrow \Gamma \cup \{\neg\varphi\}$ has a model $\;\Rightarrow\; \Gamma \nVdash \varphi$

Note: If $\Gamma \cup \{\neg\varphi\}$ is inconsistent, then $\Gamma \cup \{\neg\varphi\} \vdash \bot$ and hence $\Gamma \vdash \neg\varphi \to \bot$, i.e. $\Gamma \vdash \varphi$.

4 Constructing a Model

We need to construct a model out of something.

The model construction uses one of the two crucial ideas underlying Henkin proofs: We build a model out of syntactic material. More precisely, we equate truth in a model with membership in a maximal consistent set of sentences (MCS).

5 Maximal Consistent Sets of Sentences

A set of formulas Σ is maximal consistent iff

- Σ is consistent (i.e. $\Sigma \nvdash \bot$);
- no proper superset of Σ is consistent.

Maximal consistent sets of sentences have many nice properties:

- $\varphi \in \Sigma$ or $\neg\varphi \in \Sigma$;
- $\Sigma \vdash \varphi$ iff $\varphi \in \Sigma$;
- $\varphi \in \Sigma$ iff $\neg\varphi \notin \Sigma$;
- $\varphi \wedge \psi \in \Sigma$ iff $\varphi \in \Sigma$ and $\psi \in \Sigma$
- $\varphi \vee \psi \in \Sigma$ iff $\varphi \in \Sigma$ or $\psi \in \Sigma$
- $\varphi \to \psi \in \Sigma$ iff $\varphi \notin \Sigma$ or $\psi \in \Sigma$

6 Lindenbaum

Lindenbaum Lemma: Every consistent set of sentences Σ can be extended to a maximal consistent set of sentences Σ^*.

Construction: Assume we have an enumeration $\phi_1, \phi_2, \phi_3, \ldots$ $(n \in \omega)$ of all formulas of the language. We extend Σ step-by-step as follows:

- $\Sigma_0 = \Sigma$
- $\Sigma_{n+1} = \begin{cases} \Sigma_n \cup \{\phi_{n+1}\} & \text{if } \Sigma_n \cup \{\phi_{n+1}\} \text{ is consistent;} \\ \Sigma_n & \text{otherwise.} \end{cases}$

We define $\Sigma^* := \bigcup_{n\in\omega} \Sigma_n$.

7 Canonical Model

The canonical model for **K** is a Kripke model (W, R, V) where

- W is the set of all MCSs;
- $\Sigma R \Delta$ iff $\Delta \subseteq \{\varphi \mid \Diamond\varphi \in \Sigma\})$ (or equivalently, iff $\{\varphi \mid \Box\varphi \in \Sigma\} \subseteq \Delta$).
- $\Sigma \in V(p)$ iff $p \in \Sigma$, for all proposition letters p.

8 Existence Lemma

Existence Lemma: Let $\mathcal{M} = (W, R, V)$ be the canonical model for **K**. Suppose $\Sigma \in W$, and $\Diamond\varphi \in \Sigma$. Then there is a $\Delta \in W$ such that $\Sigma R \Delta$ and $\varphi \in \Delta$.

In order to show that Δ exists, we need to show that $\{\sigma \mid \Box\sigma \in \Sigma\} \cup \{\varphi\}$ is consistent. By Lindenbaum's Lemma, we can then extend $\{\sigma \mid \Box\sigma \in \Sigma\} \cup \{\varphi\}$ to a maximal consistent set Δ.

Assume that $\{\sigma \mid \Box\sigma \in \Sigma\} \cup \{\varphi\}$ is inconsistent.
$\Rightarrow \sigma_1 \wedge \cdots \wedge \sigma_n \wedge \varphi$ is inconsistent.
$\Rightarrow \sigma_1 \wedge \cdots \wedge \sigma_n \wedge \varphi \vdash \bot$
$\Rightarrow \sigma_1 \wedge \cdots \wedge \sigma_n \vdash \varphi \to \bot$
$\Rightarrow \sigma_1 \wedge \cdots \wedge \sigma_n \vdash \neg\varphi$.
$\Rightarrow \vdash \sigma_1 \wedge \cdots \wedge \sigma_n \to \neg\varphi$.
$\Rightarrow \vdash \Box(\sigma_1 \wedge \cdots \wedge \sigma_n \to \neg\varphi)$
$\Rightarrow \vdash \Box(\sigma_1 \wedge \cdots \wedge \sigma_n) \to \Box(\neg\varphi)$
$\Rightarrow \vdash (\Box\sigma_1 \wedge \cdots \wedge \Box\sigma_n) \to \neg\Diamond\varphi$
$\Rightarrow \neg\Diamond\varphi \in \Sigma$
$\Rightarrow \Sigma$ inconsistent   Contradiction!

9 Truth Lemma

Truth Lemma: Let $\mathcal{M} = (W, R, V)$ be the canonical model for **K**. For all worlds Σ and for all formulas φ:

$$\mathcal{M}, \Sigma \Vdash \varphi \quad \text{iff} \quad \varphi \in \Sigma.$$

Proof: Induction on the structure of φ. The interesting step is the one for the modalities:

$$\mathcal{M}, \Sigma \Vdash \Diamond\psi \quad \text{iff} \quad \Diamond\psi \in \Sigma.$$

Left-to-right: Suppose $\mathcal{M}, \Sigma \Vdash \Diamond\psi$. This means that for some Σ' such that $\Sigma R \Sigma'$, $\mathcal{M}, \Sigma' \Vdash \psi$. But then by the induction hypothesis, $\psi \in \Sigma'$. By the definition of R this implies that $\Diamond\psi \in \Sigma$.

Right-to-left: This is the interesting direction. Suppose $\Diamond\psi \in \Sigma$. We need to show that $\mathcal{M}, \Sigma \Vdash \Diamond\psi$. And this is where the Existence Lemma helps us: it tells us that there is a suitable MCS that contains ψ.

10 Completeness of the Minimal Modal Logic K

Weak completeness: $\models \varphi \;\Rightarrow\; \vdash \varphi$

Assume that $\nvdash \varphi$.
Then $\{\neg\varphi\}$ is consistent.
By Lindenbaum's Lemma, $\{\neg\varphi\}$ can be extended to a MCS Σ.
By the Truth Lemma, $\mathcal{M}, \Sigma \models \neg\varphi$.
Hence $\not\models \varphi$.

Strong completeness: $\Gamma \models \varphi \;\Rightarrow\; \Gamma \vdash \varphi$

Assume that $\Gamma \nvdash \varphi$.
Then $\Gamma \cup \{\neg\varphi\}$ is consistent.
By Lindenbaum's Lemma, $\Gamma \cup \{\neg\varphi\}$ can be extended to a MCS Σ.
By the Truth Lemma, $\mathcal{M}, \Sigma \models \Gamma$ and $\mathcal{M}, \Sigma \models \neg\varphi$.
Hence $\Gamma \not\models \varphi$.

11 Completeness of Extensions of K

In many cases, when we add additional axioms, the axioms force the canonical relation R to have the desired property. Here are some well known cases where things work that way:

Additional axioms	Canonical model
D $\Box\varphi \to \Diamond\varphi$	serial
T $\varphi \to \Diamond\varphi$	reflexive
B $\varphi \to \Box\Diamond\varphi$	symmetric
4 $\Diamond\Diamond\varphi \to \Diamond\varphi$	transitive
5 $\Diamond\varphi \to \Box\Diamond\varphi$	euclidean

It is easy to see that adding $\varphi \to \Diamond\varphi$ forces R to be reflexive. The others take a little more work.