Modal μ-Calculus Course (Tsinghua 2024)

Third homework assignment

- Deadline: 11 July, 14h20.
- Success!

Exercise 1 (bisimulation invariance)
The aim of this exercise is to give an alternative proof of the bisimulation invariance of μ-formulas, on the basis of the algebraic semantics of the modal μ-calculus. In the margin of the proof it is indicated where your input is required.

As usual, we will not make a distinction between proposition letters and variables, but we will have to define the notion of a bisimulation relative to a set P of proposition letters, see Definition 1.18 in the notes.

Theorem *Let ξ be a modal μ-formula and let P be a set of proposition letter such that $FV(\xi) \subseteq \mathsf{P}$. Then for any P-bisimulation Z linking two states s and s' in LTSs $\mathbb{S}$ and $\mathbb{S}'$, respectively, we have:*

$$\mathbb{S}, s \Vdash \xi \iff \mathbb{S}', s' \Vdash \xi.$$

Proof. The proof proceeds by induction on the complexity of ξ. You are only required to prove the inductive step for the case that ξ is of the form $\nu x.\delta$. Let P^+ be the set $\mathsf{P} \cup \{x\}$.

Assume that $Z : \mathbb{S}, s \leftrightarroweq_{\mathsf{P}} \mathbb{S}', s'$ and that $\mathbb{S}, s \Vdash \xi$. It follows immediately that there is a postfixpoint $X \subseteq S$ of the map $\delta^{\mathbb{S}}_x$ to which s belongs. Likewise, in order to prove that $\mathbb{S}', s' \Vdash \xi$ it suffices to find a single postfixpoint $X' \subseteq S'$ such that $s' \in X'$.

Let $M \subseteq S \times S$ be the *maximal* P-bisimulation on $\mathbb{S}$. (If needed, check Exercise 2.2.8 from the *Modal Logic* book for background.) Call a set $U \subseteq S$ *M-closed* if $u \in U$ and uMv imply $v \in U$. We need a special such set which is provided by the following claim.

Claim 1 There is an M-closed postfixpoint Y of δ such that $X \subseteq Y$.

Proof of Claim Let Y be the smallest M-closed subset of S such that $X \subseteq Y$. Prove that
(a) Y is a postfixpoint of δ. Hint: show that $[\![\delta]\!]^{\mathbb{S}[x \mapsto Y]}$ is M-closed. ◀

Now define $X' \subseteq S'$ to be the range of Y under Z, i.e.

$$X' := \{t' \in \mathbb{S}' \mid tZt' \text{ for some } t \in Y\}.$$

Claim 2 Z is an P^+-bisimulation between $\mathbb{S}[x \mapsto Y]$ and $\mathbb{S}'[x \mapsto X']$.

(b) PROOF OF CLAIM Provide this proof. ◀

CLAIM 3 X' is a postfixpoint of δ in $\mathbb{S}'$.

(c) PROOF OF CLAIM Provide this proof. ◀

Finally, from Claim 2 and the assumption that sZs' it immediately follows that $s' \in X'$, so by the previous claim we obtain that $\mathbb{S}', s' \Vdash \nu x.\delta$. QED

(d) What goes wrong if you try to define X' as the range of X under Z, instead of as the range of Y under Z?