Modal μ-Calculus Course (Tsinghua 2024)

Fourth homework assignment

- Deadline: 12 July, 14h20.
- **Note**: The fourth homework has three exercises, but you only have to hand in the solution to **one** exercise. Exercise 1 is relatively easy; if you hand in this one, you can get a passing grade but not the maximal grade. Exercise 2 is the main exercise: you can get full grades if you hand in a correct solution of this exercise. Exercise 3 is a bonus exercise: it is the hardest — but also the most interesting one.
- Success!

Exercise 1 Let Φ and Θ be finite sets of formulas. Prove that

$$\nabla(\Phi \cup \{\bigvee \Theta\}) \;\equiv\; \bigvee \{\nabla(\Phi \cup \Theta') \mid \varnothing \neq \Theta' \subseteq \Theta\}.$$

Exercise 2 (Small Model Property) Prove the direction (1) $\Rightarrow$ (2) of Theorem 10.10 in the Lecture Notes.

Exercise 3 (Lyndon Theorem for μML) A formula ξ is *monotone in a proposition letter* p if for all pointed models $(\mathbb{S}, s)$ and all pairs of subsets $U, U' \subseteq S$ we have

$$\text{if } U \subseteq U' \text{ then } \mathbb{S}[p \mapsto U], s \Vdash \xi \text{ implies } \mathbb{S}[p \mapsto U'], s \Vdash \xi.$$

The aim of this exercise is to prove that ξ is monotone in p iff there is a formula $\xi^M \equiv \xi$ such that ξ^M is positive in p (i.e., all occurrences of p in ξ are positive). Part (g) below forms the key part of the exercise (and is also worth most of the points).

(a) Let ξ be a clean formula that is positive in p. Show that ξ is monotone in p by means of a game-theoretic argument. (You only need to give the basic idea behind the proof – five lines should suffice.)

For the opposite direction we assume that ξ is monotone in p. Without loss of generality we may assume that ξ is disjunctive, and that ξ does not contain any subformulas of the form $\alpha \bullet \Phi$ where α is inconsistent. In particular, if $\alpha \bullet \Phi \trianglelefteq \xi$ then α cannot contain both p and $\overline{p}$.

We define the formula ξ^M by the following induction:

$$\begin{array}{lll}
x^M & := & x \\
(\bigvee \Phi)^M & := & \bigvee \Phi^M \\
(\alpha \bullet \Phi)^M & := & \alpha^M \bullet \Phi^M, \quad \text{where } \alpha^M := \alpha \setminus \{\overline{p}\}) \\
(\eta x\, \varphi)^M & := & \eta x\, \varphi^M
\end{array}$$

Here we abbreviate $\Phi^M := \{\varphi^M \mid \varphi \in \Phi\}$.

(b) Verify that ξ^M is positive in p, for every disjunctive formula ξ.

(c) Show that $\xi \models \xi^M$ (a very brief argument suffices).

Our goal will now be to show that $\xi^M \models \xi$. For this purpose, let $(\mathbb{S}, s)$ be a pointed model such that $\mathbb{S}, s \Vdash \xi^M$. We will define a subset $U \subseteq V(p)$ such that $\mathbb{S}[p \mapsto U], s \Vdash \xi$.

(d) Why does this suffice?

We may assume that $\mathbb{S}, s$ *strongly* satisfies ξ^M.

(e) Why is this without loss of generality?

Let f be some winning strategy witnessing that $\mathbb{S}, s \Vdash \xi^M$.

(f) Suppose that the position $(\beta \bullet \Psi, t)$ is f-reachable. Show that $\mathbb{S}, t \Vdash \bigwedge \beta$.

Now assume that f is a *separating* winning strategy for $\exists$ witnessing that $\mathbb{S}, s \Vdash_s \xi^M$. We define U as the set of all nodes $t \in V(p)$ such that, for some subformula $\alpha \bullet \Phi$ of ξ, the position $(\alpha^M \bullet \Phi^M, t)$ is f-reachable, while at the same time $\mathbb{S}, t \Vdash \bigwedge \alpha$.

(g) Show that $\mathbb{S}[p \mapsto U], s \Vdash \xi$.

Hint: Provide $\exists$ with a strategy f' in $\mathcal{E}_U := \mathcal{E}(\xi, \mathbb{S}[p \mapsto U])$ starting at (ξ, s) that is tightly linked to her strategy f in $\mathcal{E} := \mathcal{E}(\xi^M, \mathbb{S})$ starting at (ξ^M, s).

In order to show that f' is a winning strategy for $\exists$ in $\mathcal{E}_U$, the key claim to prove is that, whenever an f'-guided match of $\mathcal{E}_U$ reaches a position of the form $(\alpha \bullet \Phi, t)$, it holds that $\mathbb{S}[p \mapsto U], t \Vdash \bigwedge \alpha$. Here you need the condition that α does not contain both p and $\overline{p}$.