Topological Approaches to Epistemic Logic

Lecture 1: Motivation and Topological Preliminaries

Aybüke Özgün

ILLC, University of Amsterdam

Tsinghua Logic Summer School
14.07.2025

Contents

Contact Information

Course Website: http://tsinghualogic.net/JRC/topological-approaches-to-epistemic-logic/

All course material will be posted here! We will also post updates on the WeChat group.

Lecturer: Aybüke Özgün
email: a.ozgun@uva.nl
website: https://sites.google.com/site/ozgunaybuke/

Teaching Assistants:

- Yumin Ji (m2e07@naver.com)
- Wenfei Ouyang (oywf23@mails.tsinghua.edu.cn)

For questions

- For practical questions you can use WeChat.
- For content related questions, please send an email to a.ozgun@uva.nl or Yumin or Wenfei, or just ask me after the lectures.

Prerequisites

- Basic knowledge of modal logic and a reasonable level of mathematical maturity.
  - Relational/Kripke semantics for modal logic
  - Soundness & completeness
- No background in topology is necessary: all the required topological notions will be introduced in the course.
- Some knowledge of and interest in (dynamic) epistemic logic is advantageous but not required.

Course Structure

- 45 min + 5 min break + 45 min + 15 min break + 45 min.
- There will be 3 homework and a take-home exam.

Assessment

- each homework: 20% (homeworks in total = 60%)
- take-home exam: 40%

Deadlines

- Homework 1: July 16, 13h00
- Homework 2: July 17, 13h00
- Homework 3: July 18, 13h00
- Take-home exam: July 20, midnight

Homework vs Practise Questions

- Homework *is graded* and part of your final grade.
- Practise questions are *not* to be submitted or graded.
- Practise questions are for you to solve more exercises. They will be given on slides or in the handout.
- Students are encouraged to work on the practise questions together. *Homework should be done individually and independently.*

Course Materials - General Sources

- Any book on introduction to general topology. Some options:

  Engelking, R. (1989) *General Topology*, volume 6. Heldermann Verlag, Berlin, second edition.

  Dugundji, J. (1965) *Topology*. Allyn and Bacon Series in Advanced Mathematics. Prentice Hall.

- Modal Logic

  Blackburn P, Rijke M de, Venema Y. (2001) *Modal Logic*. Cambridge University Press.

- Modal Logics for Topology

  van Benthem, J. and Bezhanishvili, G. (2007). Modal logics of space. In *Handbook of Spatial Logics*, pages 217-298. Springer Verlag.

Course Materials - More...

- A number of research papers and dissertation chapters.
- I will use slides during lectures.
- Handout with the background information.

All course material will be made available on the course website.

Tentative Outline

Day 1: Motivation and Topological Preliminaries

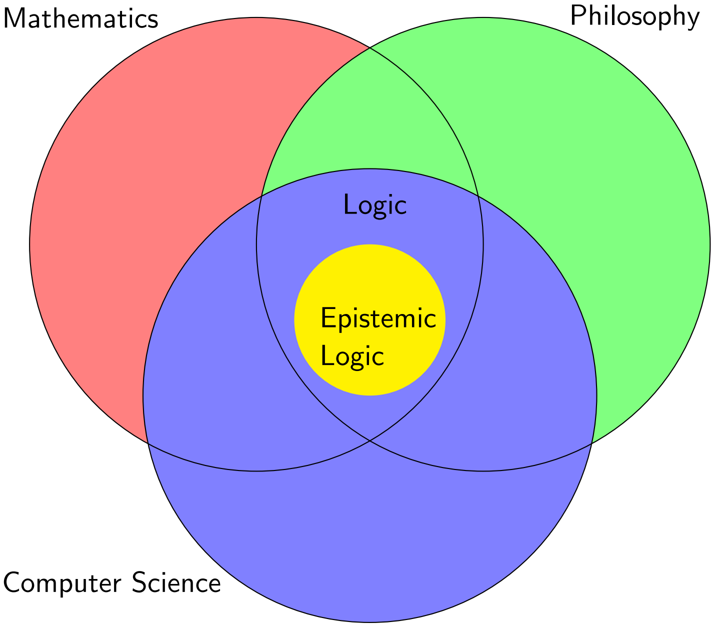
Day 2: Topological Semantics for Modal Logic

Day 3: Topological Evidence Models

Day 4: Subset Space Semantics and Topo-Logic

Day 5: Overview of Selected Topics and Summary

Contents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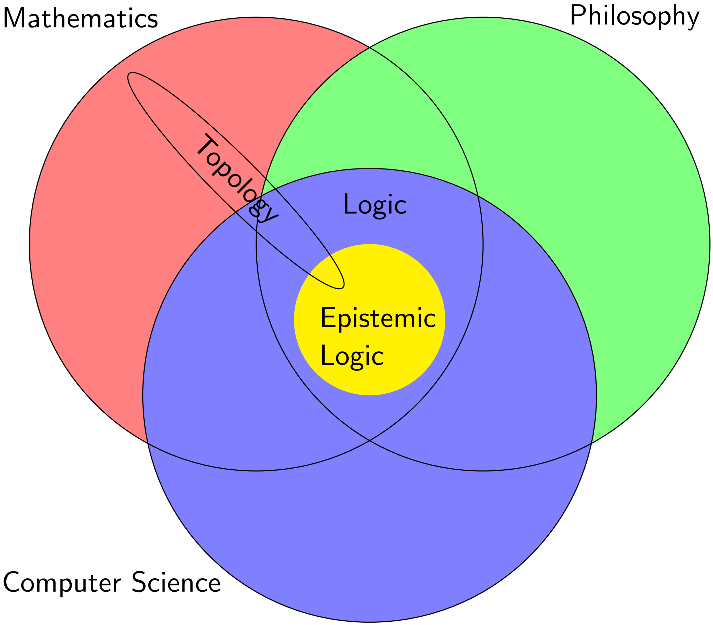
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Epistemic Logic

Epistemic Logic is an umbrella term for a variety of *(modal) logics* whose main objects of study are knowledge, belief, and related notions.[1]

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By using a modal language defined recursively as

$$\varphi ::= p \mid \neg\varphi \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \varphi \rightarrow \varphi \mid K\varphi \mid B\varphi$$

we can state various properties about the notions of knowledge, belief, and their relationships.

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Note: $\varphi \vee \psi := \neg(\neg\varphi \wedge \neg\psi)$ and $\varphi \rightarrow \psi := \neg\varphi \vee \psi$

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Epistemic Logic

$K\varphi$:= the agent knows that φ.

$B\varphi$:= the agent believes that φ.

"I know/believe that I am taller than 1.6*m."*

Epistemic Logic

For example, we can state:

$$K\varphi \to \varphi$$

"If the agent knows φ, then φ is true."

$$K\varphi \to B\varphi$$

"If the agent knows φ, then they believe φ."

$$B\varphi \to BK\varphi$$

"If the agent believes φ, then they believe that they know φ."

Epistemic Logic

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★ *Which models are appropriate?*

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- *talk only about knowledge and belief*
- *talk about evidential basis of knowledge and belief*
- *talk about awareness, limits of cognitive and computational capacities...*

Models of Epistemic Logic: Hintikka Style

The standard approach to epistemic logic formalizes knowledge and belief based on the relational possible worlds models.

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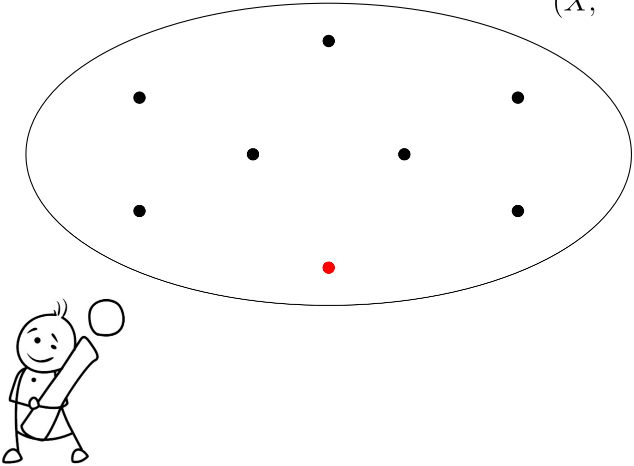

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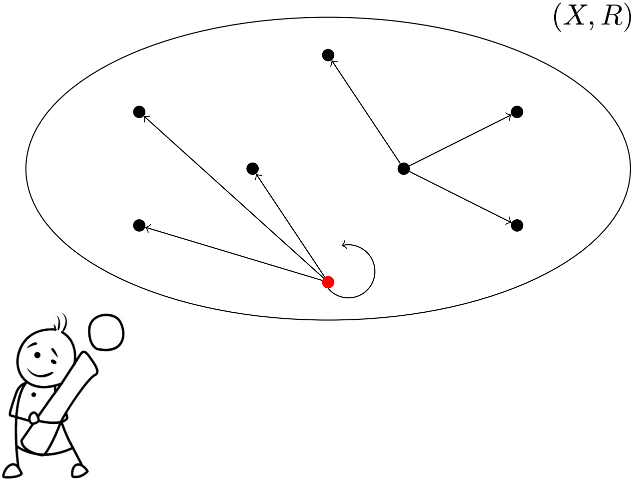

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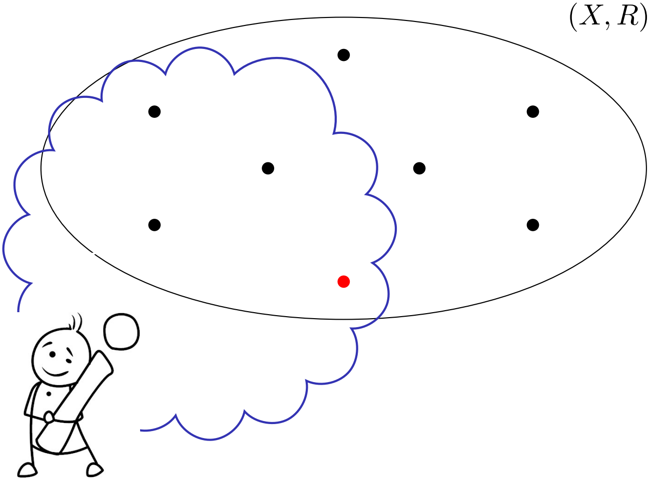

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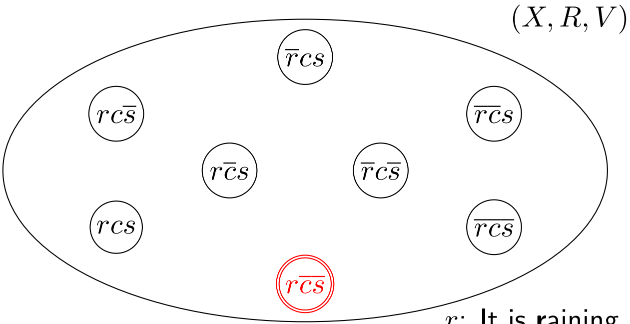

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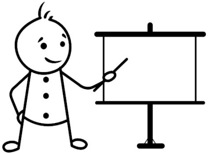
c: It is **c**old

s: Kai is **s**horter than 1.6m



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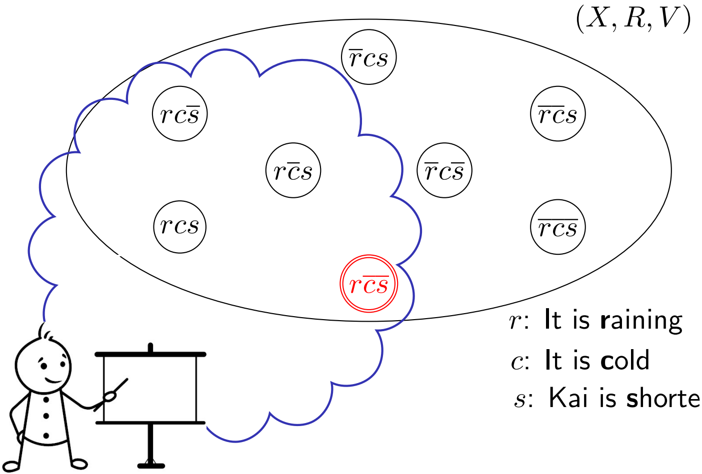

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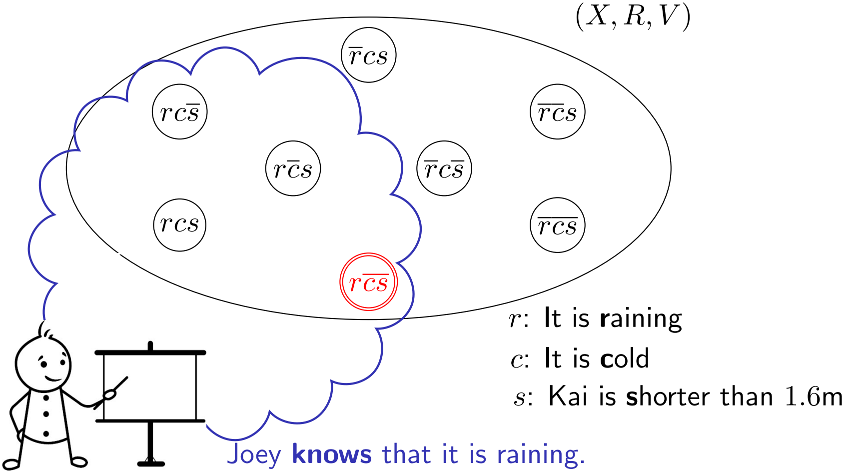

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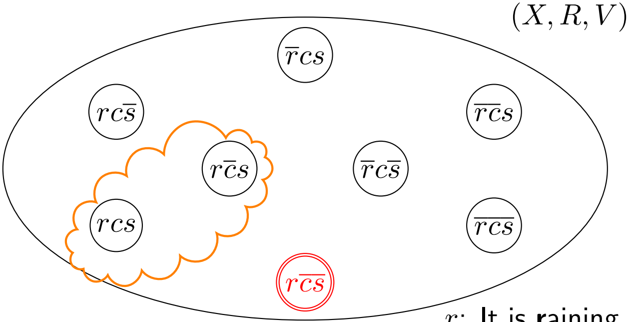

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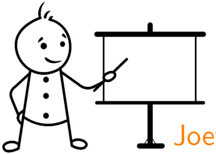

Joey **believes** that Kai is shorter than 1.6m.

Normal Epistemic Logics - Hintikka Style

Definition (Relational (Kripke) Frame/Model)

A *relational frame* is a pair $\mathcal{F} = (X, R)$, where

- X is a non-empty set of possible worlds, and
- R is a binary relation on X, i.e., $R \subseteq X \times X$.

A *relational model* is a tuple $\mathcal{M} = (X, R, V)$, where (X, R) is a Kripke frame and V : Prop $\to \mathcal{P}(X)$ is a valuation map.

Given a Kripke model $\mathcal{M} = (X, R, V)$ and a possible world $x \in X$, the pair $(\mathcal{M}, x)$ is called a *pointed model*.

Normal Epistemic Logics - Hintikka Style

Given a Kripke model $\mathcal{M} = (X, R, V)$:

- *possible worlds* represent the ways the world could be or could have been. They are *complete* and *consistent*. One of the worlds in X represents the actual world.
- R is called the *accessibility* or *indistinguishability* relation.

  $xRy :=$ *the agent cannot distinguish* y *from* x *(when in* x*).*
- $V(p)$ is the set of all possible worlds in the model where p is true.

Normal Epistemic Logics - Hintikka Style

Recall the language of epistemic logic $\mathcal{L}_K$:[2]

$$\varphi ::= p \mid \neg\varphi \mid \varphi \wedge \varphi \mid K\varphi$$

Definition (Relational Semantics)

Given a Kripke model $\mathcal{M} = (X, R, V)$ and a state $x \in X$, *truth* of a formula in the language $\mathcal{L}_K$ is defined recursively as follows:

$\mathcal{M}, x \models p$	iff	$x \in V(p)$, where $p \in$ Prop
$\mathcal{M}, x \models \neg\varphi$	iff	not $\mathcal{M}, x \models \varphi$
$\mathcal{M}, x \models \varphi \wedge \psi$	iff	$\mathcal{M}, x \models \varphi$ and $\mathcal{M}, x \models \psi$
$\mathcal{M}, x \models K\varphi$	iff	for all $y \in X$, if xRy then $\mathcal{M}, y \models \varphi$.

[2] In the handout in Section 1.1, we use $\Box$ instead of K. This is only a notational change, the frameworks are the same.

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In other words: φ is *known/believed at* w iff it is true at every possible world that is epistemically indistinguishable from w.

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Knowledge/Belief = truth in all epistemically possible worlds

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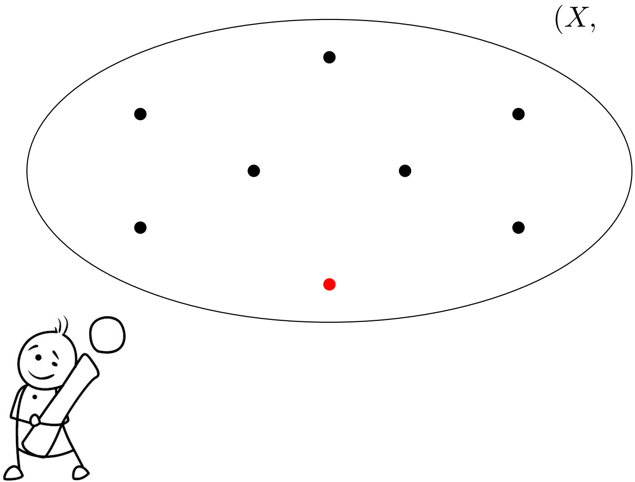
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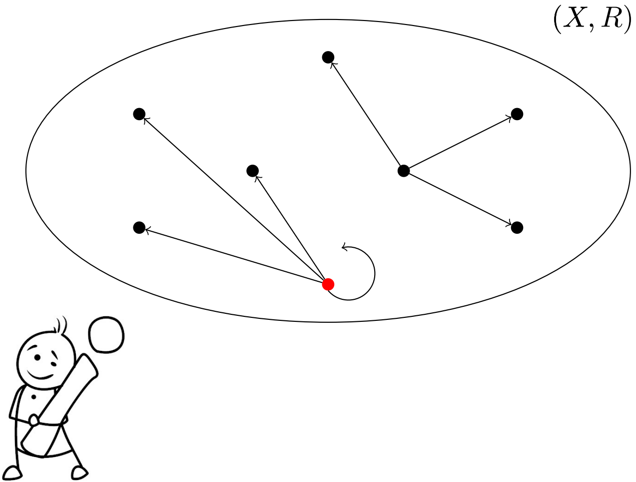
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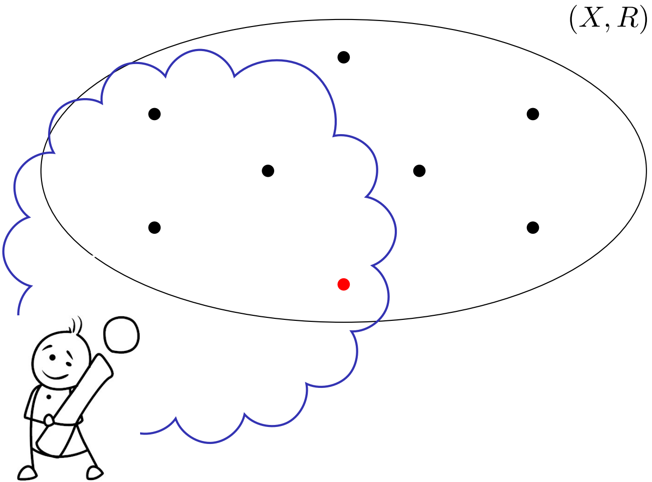
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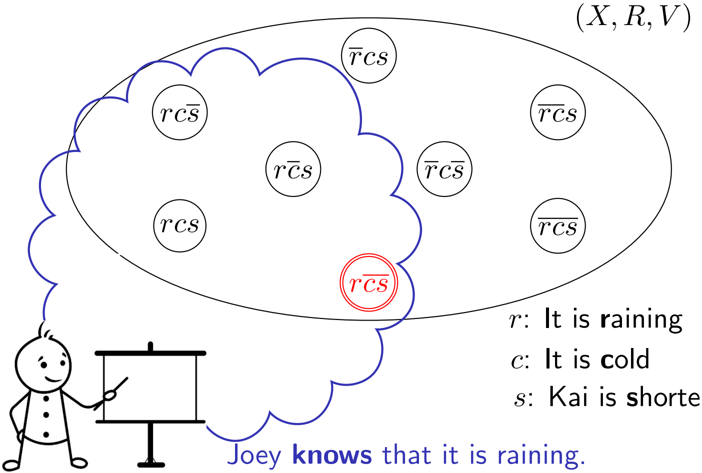

r: It is **r**aining

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s: Kai is **s**horter than 1.6m

Joey **knows** that it is raining.

Hintikka Style Epistemic Logic

Pros

(+) relatively easy

(+) well-developed model theory

(+) works well for, e.g., derivative attitudes such as

- what *one ought to know* given what one knows,
- what one can *potentially derive* from one's given body of information.

Hintikka Style Epistemic Logic

Cons

(–) not rich enough to talk about evidence (and further evidence acquisition)

★ *How to represent evidence?*

★ *How does evidence relate to knowledge, belief and justification?*

★ *How to represent learning after having acquired further evidence?*

(–) models idealized agents (without cognitive, computational, and conceptual limits)

★ *How to model non-ideal but logically competent reasoners?*

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★ *How to represent learning after having acquired further evidence?*

(–) ~~models idealized agents (without cognitive, computational, and conceptual limits)~~ ⇒ Another course!

★ ~~*How to model non-ideal but logically competent reasoners?*~~

Epistemic Logic vs. Topology

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Epistemic Logic is an umbrella term for a variety of *(modal) logics* whose main objects of study are knowledge, belief, and related notions.

Topology is the abstract mathematical study of geometric structures that are unaltered by elastic change of shapes and sizes.

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Epistemic Logic & Topology

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Epistemic Logic & Topology

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measurement/observation, argument, justification, belief, knowledge...

$\approx$

open, close, dense, nowhere dense sets, interior, closure, derivative...

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We can employ topological tools to:

- represent infallible, factive, false, misleading evidence,
- distinguish current evidence from potential evidence,
- study knowledge, knowability, and belief.
- formalize verifiability, falsifiability, and inductive learnability...

Topology as Information Structures

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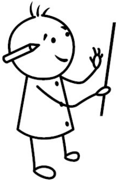
Joey believes that Kai is shorter than 1.6m,



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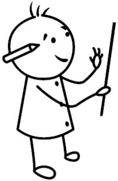

Joey believes that Kai is shorter than 1.6m, because Joey has taken many measurements with different devices: none of them contradicts Joey's claim and some support it.

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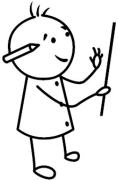

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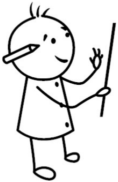

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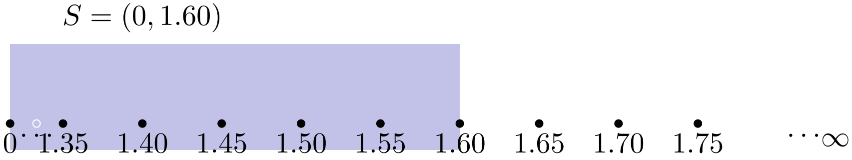

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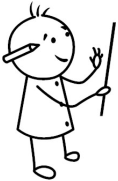

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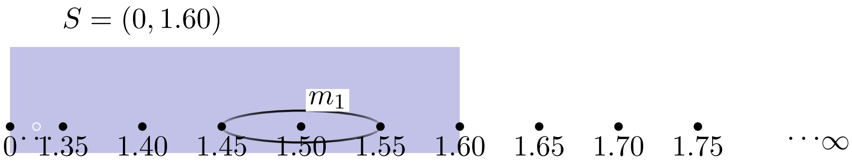

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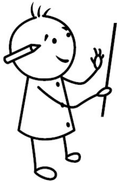

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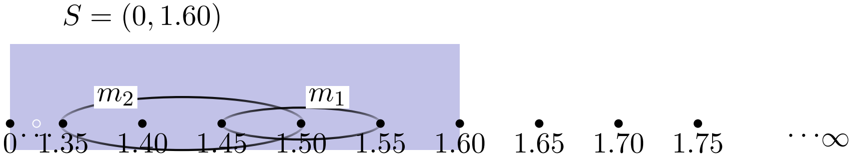

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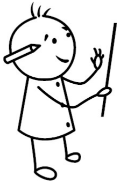

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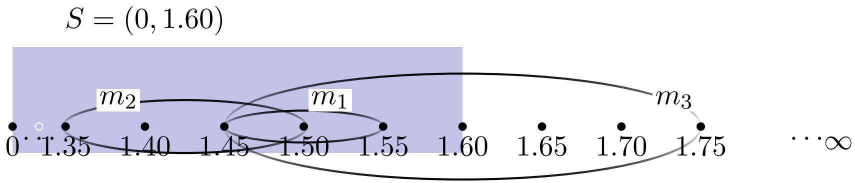
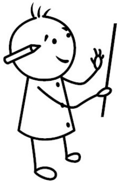


Topology as Information Structures



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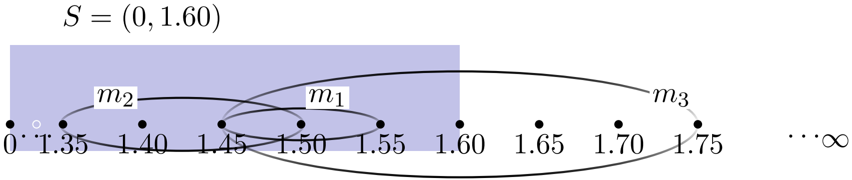

This situation can easily be formalized on the real number line $\mathbb{R}$.

Topology as Information Structures

Joey is an evolutionary biologist, investigating an animal fossil. He receives pieces of evidence from three sources (from colleagues or experiments):

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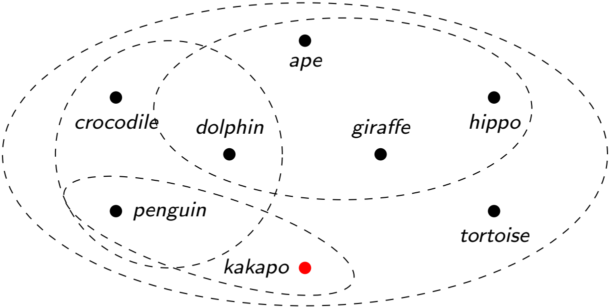

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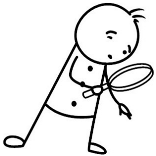

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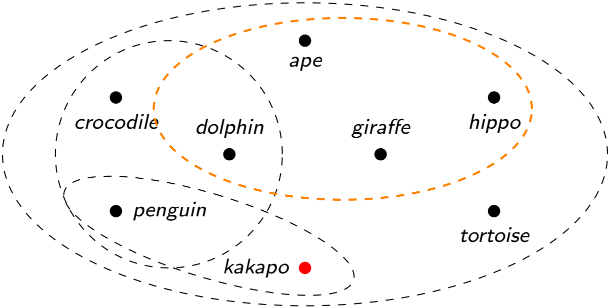

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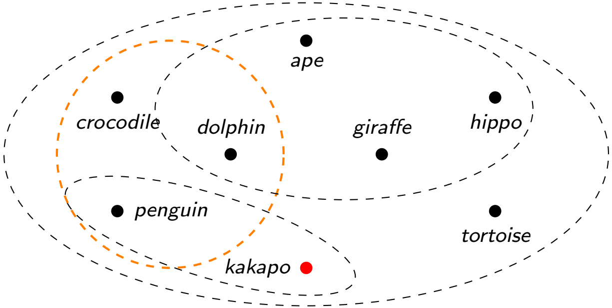

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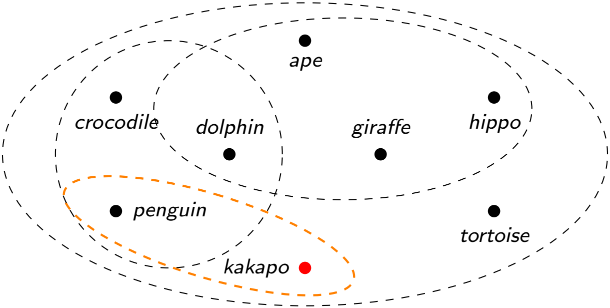

★ *How can Joey form consistent beliefs based on their evidence?*

Contents

Topological Spaces

A *topological space* is a pair (X, τ), where X is a nonempty set and $\tau \subseteq \mathcal{P}(X)$ is a family of subsets of X such that

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2. τ is closed under arbitrary unions:
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Non-example: $X = \{1, 2, 3\}$ and $\tau = \{\emptyset, \{1\}, \{2\}, \{1, 2, 3\}\}$

Why?

Topological Spaces - Terminology

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   Can you think of a set that is a clopen in all topologies?

Neighborhoods, interior points, limit points

Given a topological space (X, τ):

A(n) *(open) neighborhood* of a point $x \in X$ is an open set $U \in \tau$ with $x \in U$.

An *interior point* of a set $A \subseteq X$ is a point $x \in X$ s.t. there exists a neighborhood U of x with $U \subseteq A$.

Interior and closure

Given a topological space (X, τ) and $A \subseteq X$:

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The *interior of A* is the set of all its interior points:

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Fact 3. $Cl(A) = X \setminus Int(X \setminus A)$ for all $A \subseteq X$. (Prove this!)

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An *isolated point of A* is a point $x \in A \setminus d(A)$; in other words, a point in A that is **not** a limit point of A.

Alternative Notations

$\overline{A}$ for closure $Cl(A)$,

A' for derivative $d(A)$.

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5. In the discrete topology on $\mathbb{R}$ and any $A \subseteq \mathbb{R}$, $Cl(A) = A$ and $Int(A) = A$ for any set A (Prove this!)
6. In any discrete topology $(X, \mathcal{P}(X))$, $d(A) = \emptyset$ for all $A \subseteq X$ (Prove this!)

Properties of Int and Cl

The closure operator Cl of any topological space (X, τ) satisfies the so-called *Kuratowski properties/axioms*:

1. $Cl(\emptyset) = \emptyset$
2. $A \subseteq Cl(A)$ for all $A \subseteq X$
3. $Cl(A \cup B) = Cl(A) \cup Cl(B)$ for all $A, B \subseteq X$
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A Kuratowski closure operator is an alternative to the standard definition of topology:

for any Kuratowski closure operator $Cl : \mathcal{P}(X) \to \mathcal{P}(X)$, $\{X \setminus A \mid A = Cl(A)\}$ defines a topology on X.

Prove this!

Properties of Int and Cl

The interior satisfies the dual properties:

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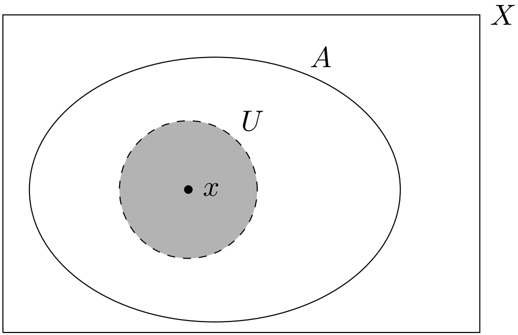
Epistemic interpretations

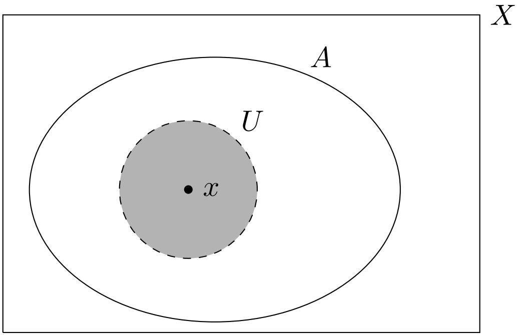


Think of U as a piece of evidence that (imperfectly) indicates the true state of the world: the points in U are precisely those that are compatible with the evidence.

E.g., U might be the result of some measurement with error.

[2] Special thanks to Adam Bjorndahl for this slide.

Epistemic interpretations



Such a "measurement" U is not precise enough to tell you the exact state of the world.

However, it can still be informative: in the above, it is precise enough to indicate that A holds.

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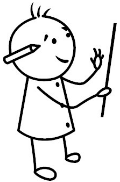
Topology as Information Structures



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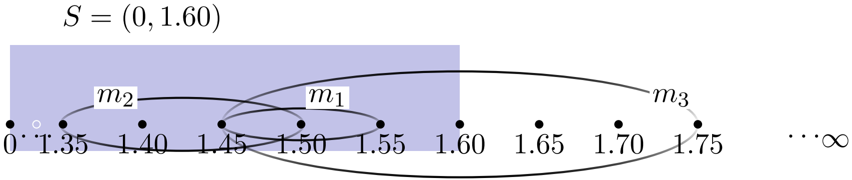

This situation can easily be formalized on the real number line $\mathbb{R}$.

Epistemic interpretations

Open sets as *verifiable properties*: one for which there is evidence, whenever it is true.

Closed sets as *falsifiable property*: one against which there is evidence, whenever it is false.

Read $Int(A)$ as *'A is known (or knowable)'* based on evidence.

Read $Cl(A)$ as *'A is epistemically possible'* (compatible with all evidence).

Epistemic interpretations

$Int(X) = X$	Tautologies are known/knowable
$Int(A) \subseteq A$	Factivity of Knowledge/ability
$Int(A \cap B) = Int(A) \cap Int(B)$	Dist. over Conjunction
$Int(Int(A)) = Int(A)$	Positive Introspection