Topological Approaches to Epistemic Logic

Lecture 2: Topological Semantics for Knowledge and Belief

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Contents

Subbasis, Basis, Generated Topology

A *subbasis* is a family $\Sigma \subseteq \mathcal{P}(X)$ of subsets of a set X s.t. for every point $x \in X$ there exists some set $O \in \Sigma$ with $x \in O$; i.e. $\bigcup \Sigma = X$.

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A *basis* (or *base*) $\mathcal{B}$ is a subbasis satisfying in addition

$$\forall B, B' \in \mathcal{B}\, \forall x \in B \cap B'\, \exists B'' \in \mathcal{B} \text{ s.t. } x \in B'' \subseteq B \cap B'.$$

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Given a subbasis $\Sigma \subseteq \mathcal{P}(X)$, the *topology τ_Σ generated by Σ on X* is the smallest topology (on X) that includes Σ.

τ_Σ consists of:

- $\emptyset$,
- X,
- Finite intersections of elements of Σ,
- Arbitrary unions of the finite intersections.

Subbasis, Basis, Generated Topology

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Subbasis: Examples

Example 1

For $X = \{0, 1, 2, 3\}$, let $\Sigma = \{\{0, 1\}, \{1, 2\}, \{1, 2, 3\}\}$.

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Example 2

The set

$$\{(-\infty, a) \mid a \in \mathbb{Q}\} \cup \{(b, \infty) \mid b \in \mathbb{Q}\}$$

is a subbasis for the standard topology on $\mathbb{R}$.

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The set

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Is this also a basis for the standard topology on $\mathbb{R}$? Justify your answer!

Basis: Examples

Example 3

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- For the standard topology on $\mathbb{R}$, $\{(a, b) \mid a < b, a, b \in \mathbb{Q}\}$ is also a basis (it is a countable basis).
- For the discrete topology on $\mathbb{R}$, $\{\{x\} \mid x \in \mathbb{R}\}$ is a basis.

Contents

Recall: Syntax of basic modal logic: $\mathcal{L}_K$

$$\varphi ::= p \mid \varphi \wedge \varphi \mid \neg\varphi \mid K\varphi$$

where $p \in \mathrm{Prop}$, a countable (or finite) set of *propositional variables*.

Note 1: We employ $\hat{K}\varphi$ as an abbreviation for $\neg K \neg \varphi$.

Note 2: $\varphi \vee \psi := \neg(\neg\varphi \wedge \neg\psi)$, $\varphi \rightarrow \psi := \neg\varphi \vee \psi$, and $\varphi \leftrightarrow \psi := (\varphi \rightarrow \psi) \wedge (\psi \rightarrow \varphi)$.

Interior semantics

Definition 4 (Topo-Model)

A *topo-model* is a tuple $\mathcal{X} = (X, \tau, V)$, where (X, τ) is a topological space and $V : Prop \to \mathcal{P}(X)$ is a *valuation* function, assigning each propositional variable to a set of points in X.

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Definition 5 (Interior semantics for $\mathcal{L}_K$)

Given a topo-model $\mathcal{X}{=}(X, \tau, V)$ and a state $x \in X$, *truth* of a formula in the langauge $\mathcal{L}_K$ is defined recursively as follows:

$\mathcal{X}, x \models p$	iff	$x \in V(p)$
$\mathcal{X}, x \models \neg\varphi$	iff	not $\mathcal{X}, x \models \varphi$
$\mathcal{X}, x \models \varphi \wedge \psi$	iff	$\mathcal{X}, x \models \varphi$ and $\mathcal{X}, x \models \psi$
$\mathcal{X}, x \models K\varphi$	iff	$(\exists U \in \tau)(x \in U$ and $\forall y \in U,\ \mathcal{X}, y \models \varphi)$

Semantic clauses for $\vee$, $\rightarrow$, and $\leftrightarrow$

Note 2: $\varphi \vee \psi := \neg(\neg\varphi \wedge \neg\psi)$, $\varphi \rightarrow \psi := \neg\varphi \vee \psi$, and $\varphi \leftrightarrow \psi := (\varphi \rightarrow \psi) \wedge (\psi \rightarrow \varphi)$.

$$
\begin{array}{lcll}
\mathcal{X}, x & \models & \varphi \vee \psi & \text{iff } \mathcal{X}, x \models \varphi \text{ or } \mathcal{X}, x \models \psi \\
\mathcal{X}, x & \models & \varphi \rightarrow \psi & \text{iff (if } \mathcal{X}, x \models \varphi \text{ then } \mathcal{X}, x \models \psi) \\
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Interior semantics ctd.

$$\mathcal{X}, x \models K\varphi \text{ iff } (\exists U \in \tau)(x \in U \text{ and } \forall y \in U, \mathcal{X}, y \models \varphi)$$

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$$\mathcal{X}, x \models K\varphi \text{ iff } (\exists U \in \tau)(x \in U \text{ and } \forall y \in U, \mathcal{X}, y \models \varphi)$$

We define the *truth set* of a formula as the set of points where the formula holds:

$$[\![\varphi]\!]^{\mathcal{X}} = \{x \in X \mid \mathcal{X}, x \models \varphi\}$$

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For $K\varphi$:

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For $\hat{K}\varphi$:

$$x \in [\![\hat{K}\varphi]\!]^{\mathcal{X}} \text{ iff } (\forall U \in \tau)(x \in U \text{ then } U \cap [\![\varphi]\!]^{\mathcal{X}} \neq \emptyset)$$

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$$[\![\hat{K}\varphi]\!]^{\mathcal{X}} = Cl([\![\varphi]\!]^{\mathcal{X}})$$

Truth and Validity

We say

- φ is true at point x (in model $\mathcal{X}$) if $\mathcal{X}, x \models \varphi$.
- φ is valid in the topo-model $\mathcal{X}$, denoted by $\mathcal{X} \models \varphi$, if $\mathcal{X}, x \models \varphi$ for all $x \in X$.
- φ is (topologically) valid if $\mathcal{X} \models \varphi$ for all topo-models $\mathcal{X}$.
- φ is (topologically) satisfiable if $\not\models \neg\varphi$; i.e. if φ is true at *some* point in *some* topo-model $\mathcal{X}$.

For a set of formulas Φ

- Φ is satisfiable if there is a point in some topo-model that makes *all* sentences in Φ true.

Epistemic interpretation

This semantics can be interpreted as an *evidence-based conception of knowledge*.

Points $x \in X$: all the *possibilities* ("possible worlds", states, descriptions of the world) that are consistent with an agent's information.

EPISTEMOLOGY	TOPOLOGY
Directly observable basic evidence	Subbasis (Σ)
Directly observable combined evidence	Basis ($\mathcal{B}$)
Verifiable evidence	Open Sets (τ)
Factive evidence at x	Open neighbourhood $U \ni x$

Epistemic interpretation

In fact, it would be natural to require the family $\mathcal{B}$ of directly observable evidence to have *stronger closure properties*:

$$X \in \mathcal{B}$$

(*tautologies are directly observable*), and

$$A, B \in \mathcal{B} \Rightarrow A \cap B \in \mathcal{B}$$

(*we can cumulate observations*).

These properties imply that $\mathcal{B}$ is a basis.

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$$\text{iff } (\exists U \in \mathcal{B})(x \in U \text{ and } U \subseteq [\![\varphi]\!]^{\mathcal{X}})$$

In words:

φ is "*known*" or "*knowable*"

iff

the agent *has correct evidence for* φ

iff

there is some *some directly observable true evidence supporting* φ.

Intermezzo: Link to the Relational Semantics

There is a tight link between the reflexive and transitive Kripke frames (preordered sets) and Alexandroff spaces.

Alexandroff Topologies

A topological space (X, τ) is an *Alexandroff space* if τ is closed under arbitrary intersections, i.e., $\bigcap \mathcal{A} \in \tau$ for any $\mathcal{A} \subseteq \tau$.

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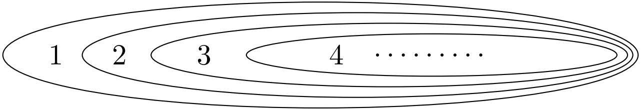
Example 6

On $\mathbb{N}$, let $\tau = \{\{m \mid m \geq n\} \mid \text{for some } n \in \mathbb{N}\} \cup \{\emptyset\}$.



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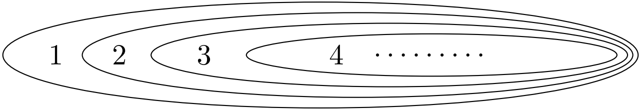

Non-Example 1

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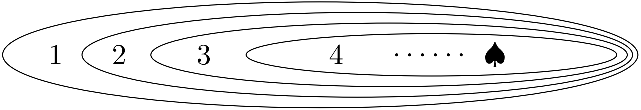

From preorders to topologies

Let (X, R) be a preordered set. Then, the set

$$\tau_R = \{A \mid A \text{ is a upward closed set}^1 \text{ of } (X, R)\}$$

is a topological space. We call τ_R is the *upset topology on the preordered set* (X, R).

[1] A is called an *upward-closed set* (or, in short, *up-set*) of (X, R) if for each $x, y \in X$, xRy and $x \in A$ imply $y \in A$.

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Fact 1. *Every upset topology is an Alexandroff topology*.

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From topologies to preorders

Specialization Preorder - $\sqsubseteq$

Give a topological space (X, τ) and two points $x, y \in X$, we say that *x is a specialization of y, $x \sqsubseteq_\tau y$,* if every (open) neighborhood of x is also a neighborhood of y:

$$x \sqsubseteq_\tau y \quad \text{iff} \quad \forall U \in \tau(x \in U \Rightarrow y \in U).$$

If we are given a subbasis Σ for the topology τ, then it is easy to see that we also have:

$$x \sqsubseteq_\tau y \quad \text{iff} \quad \forall U \in \Sigma(x \in U \Rightarrow y \in U).$$

From topologies to preorders

Specialization Preorder - $\sqsubseteq$

Equivalently:

$x \sqsubseteq_\tau y$ iff x is contained in every *closed* set containing y.

Here are some more equivalent characterizations:

$$x \sqsubseteq_\tau y \quad \text{iff} \quad x \in Cl(\{y\}) \quad \text{iff} \quad Cl(\{x\}) \subseteq Cl(\{y\}).$$

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$\sqsubseteq_\tau$ is a preorder - reflexive and transitive relation - on X, called the *specialization preorder*.

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$\sqsubseteq_\tau$ is a preorder - reflexive and transitive relation - on X, called the *specialization preorder*.

Fact 2. *Every open set is upwards-closed wrt the specialization preorder.*

Alexandroff Spaces are preordered sets!

Given a preordered set (X, R),

$$R = \sqsubseteq_{\tau_R} .$$

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Given a topological space (X, τ),

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$$\tau \subseteq \tau_{\sqsubseteq_\tau} .$$

(X, τ) is an Alexandroff space iff $\tau = \tau_{\sqsubseteq_\tau}$.

Link to the Relational Semantics - A special case

When the underlying topology is Alexandroff given by the upsets wrt to a given preorder R on X, that is, τ_R, our topological semantics coincides with the standard relational semantics.

Recall:

An *$S4$-Kripke frame* is just a pair (X, R), consisting of a set X of possible worlds and a preorder (=reflexive and transitive relation) R on X.

An *$S4$-Kripke model* is a triplet $\mathcal{M} = (X, R, V)$, where (X, R) is an $S4$ frame and $V : Prop \to \mathcal{P}(W)$ is a valuation.

Recall: Relational semantics

Given a relational model (X, R, V) and $x \in X$, the relational semantics for $\mathcal{L}_K$ is defined recursively as

$$
\begin{array}{llll}
\mathcal{X}, x & \models & p & \text{iff } x \in V(p) \\
\mathcal{X}, x & \models & \varphi \wedge \psi & \text{iff } \mathcal{X}, x \models \varphi \text{ and } \mathcal{X}, x \models \psi \\
\mathcal{X}, x & \models & \neg\varphi & \text{iff } \mathcal{X}, x \not\models \varphi \\
\mathcal{X}, x & \models & K\varphi & \text{iff } (\forall y \in X)(xRy \text{ then } \mathcal{X}, y \models \varphi)
\end{array}
$$

Correspondence between relational and topological models

For an $S4$-Kripke model $\mathcal{M} = (X, R, V)$, let $B(\mathcal{M}) = (X, \tau_R, V)$ - a topo-model.

For Alexandroff model $\mathcal{X} = (X, \tau, V)$, let $A(\mathcal{X}) = (X, \sqsubseteq_\tau, V)$ - an $S4$-Kripke model.

Correspondence between relational and topological models

Proposition 7

For all $\varphi \in \mathcal{L}_K$,

1. *for any* $S4$*-Kripke model* $\mathcal{M} = (X, R, V)$ *and* $x \in X$,

$$\mathcal{M}, x \models \varphi \text{ iff } B(\mathcal{M}), x \models \varphi;$$

2. *for any Alexandroff model* $\mathcal{X} = (X, \tau, V)$ *and* $x \in X$,

$$\mathcal{X}, x \models \varphi \text{ iff } A(\mathcal{X}), x \models \varphi.$$

Soundness and Completeness Results

	$\mathsf{S4}_K$ axioms	Kuratowski axioms
(K_K)	$K(\varphi \wedge \psi) \leftrightarrow (K\varphi \wedge K\psi)$	$Int(A \cap B) = Int(A) \cap Int(B)$
(T_K)	$K\varphi \rightarrow \varphi$	$Int(A) \subseteq A$
$(\mathsf{4}_K)$	$K\varphi \rightarrow KK\varphi$	$Int(A) \subseteq Int(Int(A))$
(Nec)	from φ, infer $K\varphi$	$Int(X) = X$

Table: $\mathsf{S4}_K$ vs. Kuratowski axioms

Soundness and Completeness Results

	$\mathsf{S4}_K$ axioms	Kuratowski axioms
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(Nec)	from φ, infer $K\varphi$	$Int(X) = X$

Table: $\mathsf{S4}_K$ vs. Kuratowski axioms

Theorem 8 ([McKinsey and Tarski, 1944])

$\mathsf{S4}_K$ *is sound and complete with respect to the class of all topological spaces (under the interior semantics).*

Soundness and Completeness Results

Proposition 9 ([van Benthem and Bezhanishvili, 2007])

Every normal extension of S4$_K$ *(over the language* $\mathcal{L}_K$*) that is complete with respect to the standard relational semantics is also complete with respect to the interior semantics.*

Soundness and Completeness Results

Proposition 9 ([van Benthem and Bezhanishvili, 2007])

Every normal extension of $\mathsf{S4}_K$ *(over the language* $\mathcal{L}_K$*) that is complete with respect to the standard relational semantics is also complete with respect to the interior semantics.*

Proof.

Let L_K be a normal extension of $\mathsf{S4}_K$ that is complete with respect to the relational semantics and $\varphi \in \mathcal{L}_K$ such that $\varphi \notin \mathsf{L}_K$. Then, by relational completeness of L_K, there exists a relational model $\mathcal{M} = (X, R, V)$ and $x \in X$ such that $\mathcal{M}, x \not\models \varphi$. Since L_K extends the system $\mathsf{S4}_K$, which is complete with respect to reflexive and transitive Kripke models, R can be assumed to be at least reflexive and transitive. Then we obtain $B(\mathcal{M}), x \not\models \varphi$ (by Proposition 7). □

Soundness and Completeness Results

Definition 10

A topological space (X, τ) is called *extremally disconnected* if the closure of each open subset of X is open.

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Theorem 11 ([Gabelaia, 2001])

S4.2$_K$ *is sound and complete with respect to the class of extremally disconnected topological spaces under the interior semantics.*

Soundness and Completeness Results

Definition 12

A topological space (X, τ) is called *hereditarily extremally disconnected* if every subspace of (X, τ) is extremally disconnected.

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Theorem 13 ([Bezhanishvili et al., 2015])

$\mathsf{S4.3}_K$ *is sound and complete with respect to the class of hereditarily extremally disconnected topological spaces under the interior semantics.*

The Motivation behind *Knowledge as Interior*

According to the interior semantics, given a topo-model $\mathcal{X} = (X, \tau, V)$, we have

$$[\![K\varphi]\!] = Int([\![\varphi]\!])$$

and

$$[\![\hat{K}\varphi]\!] = Cl([\![\varphi]\!])$$

The Motivation behind *Knowledge as Interior*

According to the interior semantics, given a topo-model $\mathcal{X} = (X, \tau, V)$, we have

$$[\![K\varphi]\!] = Int([\![\varphi]\!])$$

and

$$[\![\hat{K}\varphi]\!] = Cl([\![\varphi]\!])$$

1. The interior semantics is naturally epistemic and extends the relational semantics.
2. We can talk about evidence: evidence as open sets.

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Recall:

	S4$_K$ axioms	Kuratowski axioms
(K$_K$)	$K(\varphi \wedge \psi) \leftrightarrow (K\varphi \wedge K\psi)$	$Int(A \cap B) = Int(A) \cap Int(B)$
(T$_K$)	$K\varphi \rightarrow \varphi$	$Int(A) \subseteq A$
(4$_K$)	$K\varphi \rightarrow KK\varphi$	$Int(A) \subseteq Int(Int(A))$
(Nec)	from φ, infer $K\varphi$	$Int(X) = X$

Table: S4$_K$ vs. Kuratowski axioms

Recall Theorem 8:

S4$_K$ is sound and complete with respect to the class of all topological spaces (under the interior semantics).

The Motivation behind *Knowledge as Interior*

The interior semantics is naturally epistemic and extends the relational semantics.

So, in its most general form, topologically modelled knowledge (as the interior operator) is *Factive*,

$$K\varphi \to \varphi,$$

and *Positively Introspective*,

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however, it does not necessarily possess stronger properties.

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- The interior semantics is naturally epistemic since the most general class of spaces constitutes the class of models of arguably the weakest, yet philosophically the most accepted normal system $\mathsf{S4}_K$.

Q. Is this a limitation (especially compared to the relational semantics)? Not really!

The Motivation behind *Knowledge as Interior*

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Recall:

$\mathsf{S4}_K$	The logic of all topological spaces
$\mathsf{S4.2}_K$	The logic of extremally disconnected topological spaces
$\mathsf{S4.3}_K$	The logic of hereditarily ext. disc. topological spaces
$\mathsf{S5}_K$	The logic of topological spaces whose every closed subset is open

Table: Logics of $\mathcal{L}_K$ under the interior semantics.

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Table: Logics of $\mathcal{L}_K$ under the interior semantics.

- Topological spaces provide *sufficiently flexible* structures to study knowledge of different strength.
- Moreover, the interior semantics generalizes the standard Kripke semantics for normal extensions of $\mathsf{S4}_K$.

Further extensions

- Multi-agents

  van Benthem, J., Bezhanishvilli, G., ten Cate, B., & Sarenac, D. (2005). *Modal logics for products of topologies.* Studia Logica, 84(3)(369-392).

- Common Knowledge

  Barwise, J. (1988). *Three views of common knowledge*. In Proceedings of the 2nd Conference on Theoretical Aspects of Reasoning about Knolwedge, (pp. 365-379).

  van Benthem, J., & Sarenac, D. (2004). *The geometry of knowledge*. In Aspects of universal Logic, vol. 17, (pp. 1-31).

- Logics of learning and observational effort - Subset Space Logics

  Moss, L. S., & Parikh, R. (1992). *Topological reasoning and the logic of knowledge*. In Proceedings of 4th TARK, (pp. 95-105).

- Topological versions of dynamic epistemic logic

  Zvesper, J. (2010). *Playing with Information*. Ph.D. thesis, University of Amsterdam.

Contents

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Q. Can topological semantics also account for notions of *(evidentially) justified belief* that **work well** with the previous notion of knowledge?

Interaction Between Knowledge and Belief

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Platonic equation:

$$\text{knowledge} = \text{justified true belief (JTB)} \quad + \quad (??)$$

"an agent knows φ iff φ is true, they believe that it is true and they are justified in believing that φ."

This interpretation was shattered by *Gettier's famous counterexamples* [Gettier, 1963].

A Gettier-Type Counterexample

Suppose that I have strong evidence for the proposition:

(a) Sophia owns a Ford.

My evidence might be that Sophia has at all times in the past, as far as I remember, owned a car, and always a Ford, and that she has just offered me a ride while driving a Ford. (Unbeknownst to me, it was in fact a rental car.)

I have another friend, Fernando, and I had no idea about where Fernando was last week. On the basis of (a), I believe that

(b) Sophia owns a Ford or Fernando was in Beijing last week.

I am thereby justified in believing (b). As it turns out, unbeknownst to me, Fernando was indeed in Beijing last week. Therefore, my justified belief in (b) is true.

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Is this really knowledge OR mere coincidence?

Solutions to Gettier's Challenge

The solutions can be classified in two categories:

(1) the ones that start with the weakest notion (true justified, or justifiable, belief) and adding some "missing ingredient" to the Platonic equation, to obtain "knowledge"

$$\text{JTB} + \text{X};$$

(2) the ones that start from a chosen notion of knowledge, and weaken it to obtain a "good" notion of belief.

Belief on Topological Spaces?

Q. What is the relationship between belief, evidence, and knowledge?

Q. Can topological semantics also account for notions of *(evidentially) justified belief* that **work well** with the previous notion of knowledge?

Some earlier proposals

- Belief as *the co-derivative*

  Steinsvold, C. (2006). *Topological models of belief logics*. PhD thesis, City University of New York.

- Belief as *the closure of the interior*

  Baltag, A., Bezhanishvili, N., Özgün, A., and Smets, S. (2019) *A Topological Approach to Full Belief*. Journal of Philosophical Logic, pp. 205-244.

We focus on:

- Belief as *dense interior*

  Baltag, A., Bezhanishvili, N., Özgün, A., and Smets, S. (2020) *Justifed belief, knowledge, and the topology of evidence*. Synthese **200**, 512.

  Özgün, A (2017). *Evidence in Epistemic Logic: A topological perspective*. PhD thesis. Université de Lorraine & University of Amsterdam - Chapter 5.

Steinsvold's Belief as *the co-derivative*

Recall: Given a topological space (X, τ) and $A \subseteq X$

$$x \in d(A) \text{ iff } \forall U \in \tau(x \in U \text{ implies } A \cap (U \setminus \{x\}) \neq \emptyset)$$

$$x \in t(A) \text{ iff } \exists U \in \tau(x \in U \text{ and } U \setminus \{x\} \subseteq A)$$

Note: $t(A) = X \setminus (d(X \setminus A))$

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One of the crucial properties that distinguishes knowledge from belief is its *factivity*. Steinsvold's belief is **not** necessarily factive.

[2] dense-in-itself space (i.e., a space without singleton opens) in which every derived set $d(A)$ is open.

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± $KD45_B$ is the logic of DSO-spaces[2].

- it can easily be "**Gettierized**":

$$K\varphi := B\varphi \wedge \varphi$$

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Belief as *the closure of the interior* - Motivation

We are now in a very unusual situation: belief, rather than knowledge, is the main mystery, in the topological semantics.

Q. Given the interior-based topological semantics for knowledge, how can we construct a topological semantics for belief which sits well with the *knowledge as interior*, thus can help us understand the relation between knowledge and belief?

Belief as *the closure of the interior* - Stalnaker's system

Stalnaker (2006) has proposed a logic intended to capture the relationship between knowledge and belief, where belief is interpreted in the strong sense of *subjective certainty*.

$$(\mathcal{L}_{K,B}) \quad \varphi ::= p \,|\, \neg\varphi \,|\, \varphi \wedge \psi \,|\, K\varphi \,|\, B\varphi$$

This logic extends the classic S4 system for knowledge...

(K_K)	$K(\varphi \to \psi) \to (K\varphi \to K\psi)$	Distribution
(T_K)	$K\varphi \to \varphi$	Factivity
(4_K)	$K\varphi \to KK\varphi$	Positive introspection
(Nec_K)	from φ infer $K\varphi$	Necessitation

Table: $\mathsf{S4}_K$ axioms for knowledge

Belief as *the closure of the interior* - Stalnaker's system

...with the following additional axioms.

(D_B)	$B\varphi \to \neg B \neg \varphi$	Consistency of belief
(sPI)	$B\varphi \to KB\varphi$	Strong positive introspection
(sNI)	$\neg B\varphi \to K\neg B\varphi$	Strong negative introspection
(KB)	$K\varphi \to B\varphi$	Knowledge implies belief
(FB)	$B\varphi \to BK\varphi$	Full belief

Table: Stalnaker's additional axioms

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Table: Stalnaker's additional axioms

Belief as *subjective certainty*: an agent who feels certain that φ is true also feels certain that she *knows* that φ is true.

Belief as *the closure of the interior* - Stalnaker's system

In this system, one can prove the following striking equivalence:

$$B\varphi \leftrightarrow \hat{K}K\varphi,$$

where $\hat{K}$ abbreviates $\neg K \neg$.

- Belief is equivalent to "the epistemic possibility of knowledge".
- In particular, belief can be *defined* in terms of knowledge—once you have knowledge, you get belief for free.

Recall:

$$\begin{array}{rcl} [\![K\varphi]\!] & = & Int([\![\varphi]\!]) \\ [\![\hat{K}\varphi]\!] & = & Cl([\![\varphi]\!]). \end{array}$$

We then obtain that

$$[\![B\varphi]\!] = Cl(Int([\![\varphi]\!])).$$

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Theorem 14 ([Baltag et al., 2019])

Stal *is sound and complete with respect to the class of extremally disconnected spaces.*

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Example 2: Alexandroff spaces constructed from *directed preorder*.

Belief as *the closure of the interior* - Stalnaker's system

Moreover, Stalnaker's system entails

- $KD45_B$ as the logic of belief

$$\begin{array}{|c|}\hline B(\varphi \to \psi) \to (B\varphi \to B\psi) \\ B\varphi \to \neg B \neg \varphi \\ B\varphi \to BB\varphi \\ \neg B\varphi \to B\neg B\varphi \\ \hline \end{array}$$

- $S4.2_K$ as the logic of knowledge

$$\begin{array}{|c|}\hline K(\varphi \to \psi) \to (K\varphi \to K\psi) \\ K\varphi \to \varphi \\ K\varphi \to KK\varphi \\ \hat{K}K\varphi \to K\hat{K}\varphi \\ \hline \end{array}$$

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Theorem 15 ([Gabelaia, 2001])

$S4.2_K$ *is sound and complete with respect to the class of extremally disconnected spaces (under the interior semantics).*

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Theorem 16 ([Baltag et al., 2019])

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Belief as *the closure of the interior* - Further Comments

However, we might want to work with a larger classes of topological spaces that includes more natural topological spaces.

Belief as *the closure of the interior* - Further Comments

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Q. Is the best epistemic interpretation of the interior operator knowledge? Can we give it a “more direct” epistemic reading? [Bjorndahl and Özgün, 2020, Baltag et al., 2022]

Belief as *the closure of the interior* - Further Comments

The connection between evidence and open sets comes to exist at the most elementary level, namely at the level of a subbasis.

EPISTEMOLOGY	TOPOLOGY
Directly observable basic evidence	Subbasis (Σ)
Directly observable combined evidence	Basis $(\mathcal{B})$
Verifiable evidence	Open Sets (τ)
Factive evidence at x	Open neighbourhood $U \ni x$

The interior semantics (over $\mathcal{L}_K$) is clearly not expressive enough to distinguish different types of open sets, and, in turn, cannot account for different notions of evidence possession.

Belief as *the closure of the interior* - Further Comments

The current framework does not have any syntactic representation of evidence: everything we can say about evidence has to be said at a purely semantic level.

These motivate another topological framework inspired by the evidence models introduced in [van Benthem and Pacuit, 2011].

Next Lecture: *Belief as dense interior*

Baltag, A., Bezhanishvili, N., Özgün, A., and Smets, S. (2020)
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Baltag, A., Bezhanishvili, N., Özgün, A., and Smets, S. (2015).
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