Topological Approaches to Epistemic Logic

Lecture 3: The Topology of Actual Evidence

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Motivation

We are interested in studying notions of belief and knowledge, for a rational agent who is in possession of some (possibly false, possibly mutually contradictory) pieces of evidence.

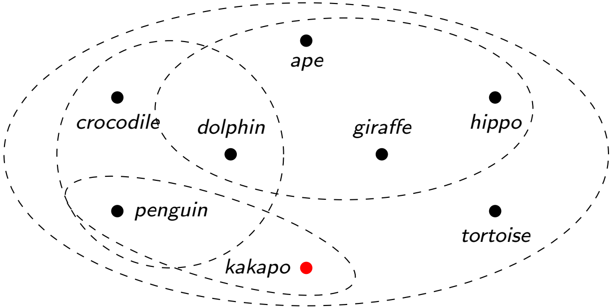
Topology as Information Structures

Joey is an evolutionary biologist, investigating an animal fossil. They receive pieces of evidence from three sources (from colleagues or experiments):

e_1: it is a mammal

e_2: it can swim

e_3: it is a non-flying bird



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- *an agent's rational belief is based on the available evidence;*
- *evidence is represented both semantically and syntactically;*
- *belief and knowledge are not primitive, they are built from evidence pieces.*

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This lecture is based on the material in [Baltag et al., 2022] and [Özgün, 2017, Chapter 5]. Relevant proofs can be found in these sources.

Preliminaries: Evidence Models

Definition ([van Benthem and Pacuit, 2011])

A *(uniform) evidence model* is a tuple $\mathcal{M} = (X, \mathcal{E}_0, V)$, where

- $X \neq \emptyset$ is the set of *possible worlds* (or "*states*");
- $\mathcal{E}_0 \subseteq \mathcal{P}(X)$ is the set of *basic evidence sets* (also called "*pieces of evidence*"), satisfying $X \in \mathcal{E}_0$ and $\emptyset \notin \mathcal{E}_0$;
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An evidence e is *factive* (or "correct") at world x if $x \in e$.

Forming Beliefs based on (Fallible) Evidence

The main idea behind van Benthem & Pacuit's semantics:

The rational agent tries to form consistent beliefs, by looking at all maximally finitely-consistent "blocks" of evidence, and believing whatever is entailed by all of them.

- "Having evidence for φ need *not* imply belief."
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Bodies of Evidence

A *body of evidence* is a *family* $F \subseteq \mathcal{E}_0$ *of evidence pieces s.t. every finitely many of them are mutually consistent*:

$$(\forall F' \subseteq_{fin} F)(F' \neq \emptyset \Rightarrow \bigcap F' \neq \emptyset)$$

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Notation:

- $\mathcal{F}$:= the family of *all bodies of evidence* over $\mathcal{M}$.
- $\mathcal{F}^{fin}$:= the family of all *finite bodies of evidence* over $\mathcal{M}$.

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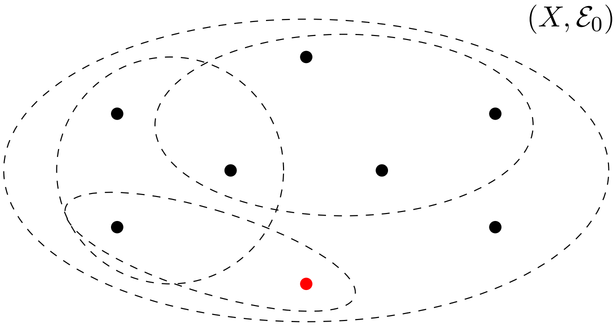
$e \in \mathcal{E}$: *direct* evidence obtained by combining finitely many pieces of direct evidence.

Evidence Models



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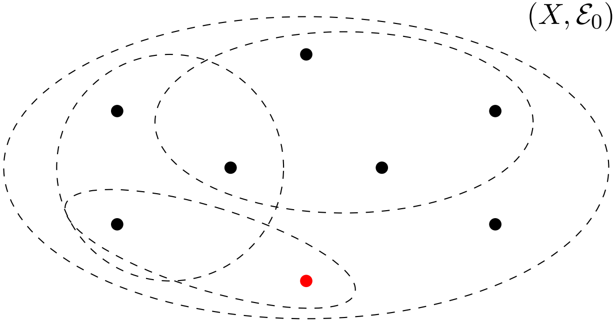


$e_0 \in \mathcal{E}_0$: a piece of *direct evidence*.

Evidence Models

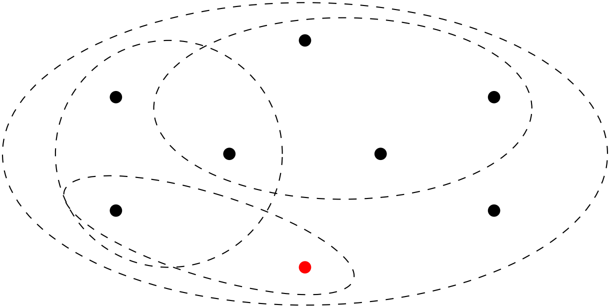
$$(X, \mathcal{E}_0)$$

$$\mathcal{E}_0 \xrightarrow[\text{finite intersections}]{} \mathcal{E}$$

$$\Cup \qquad\qquad \Cup$$

$$e_0 \qquad\qquad e$$

direct evidence *combined evidence*



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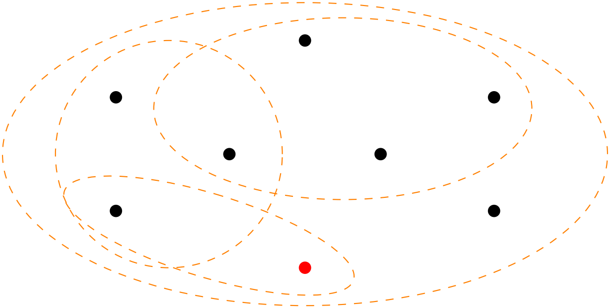

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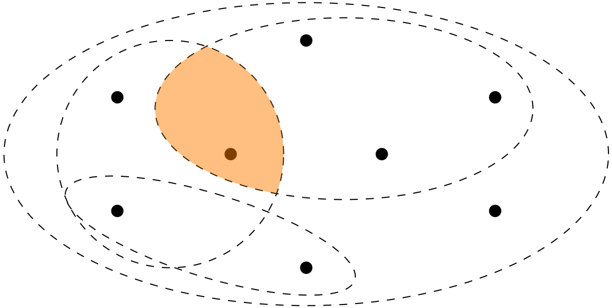

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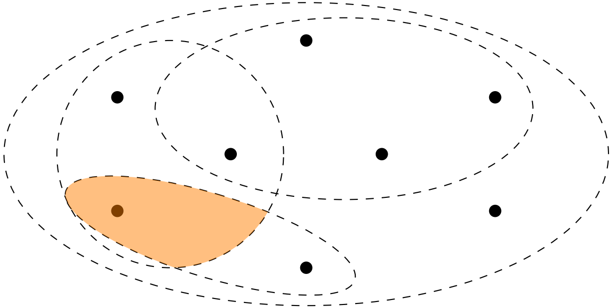

Evidential Support and Strength Order

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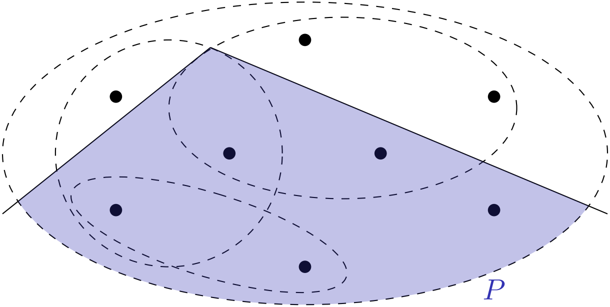
- strength order $\subseteq$ on $\mathcal{F}$:

$$F \subseteq F' := F' \textit{ is at least as strong as } F$$

- strength order $\supseteq$ on $\mathcal{E}$:

$$e \supseteq e' := e' \textit{ is at least as strong as } e$$

Evidential Support



e_0 is *basic evidence for* P iff $e_0 \subseteq P$

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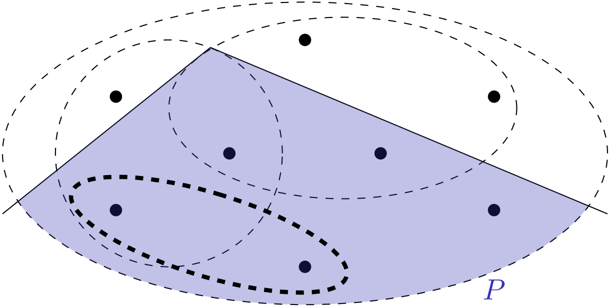

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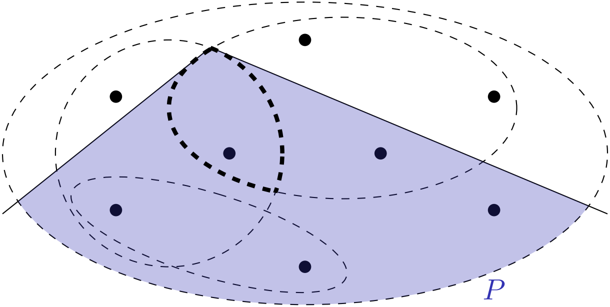

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Maximal bodies of evidence

The family of "*strongest bodies of evidence*" (maximal wrt $\subseteq$):

$$Max_{\subseteq}(\mathcal{F}) \;:=\; \{F \in \mathcal{F} \mid \forall F' \in \mathcal{F}\,(F \subseteq F' \Rightarrow F = F')\}$$

Observation: $Max_{\subseteq}(\mathcal{F}) \neq \emptyset$ (Zorn's Lemma)[1].

[1] Every partially ordered set $\mathcal{F}$ that has the property that every chain in $\mathcal{F}$ has an upper bound in $\mathcal{F}$, contains at least one maximal element.

Evidential Plausibility Order on States

The *evidential plausibility order* $\sqsubseteq_{\mathcal{E}}$ associated to an evidence model is defined by :

$$x \sqsubseteq_{\mathcal{E}} y \;\text{ iff }\; \forall e \in \mathcal{E}_0 \,(x \in e \Rightarrow y \in e)$$

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We denote the strict order by

$$x \sqsubset_{\mathcal{E}} y \text{ iff } x \sqsubseteq_{\mathcal{E}} y \text{ and } y \not\sqsubseteq_{\mathcal{E}} x.$$

The set of "most plausible worlds" (maximal worlds wrt $\sqsubseteq_{\mathcal{E}}$):

$$Max_{\sqsubseteq_{\mathcal{E}}} X \; := \; \{y \in X \mid \forall z \in X (y \not\sqsubset_{\mathcal{E}} z)\}$$

The Logic of Evidence, Belief and Infallible Knowledge

Syntax of [van Benthem and Pacuit, 2011]

$$\mathcal{L}_0 := p \mid \neg\varphi \mid \varphi \wedge \varphi \mid E_0\varphi \mid B\varphi \mid [\forall]\varphi$$

$E_0\varphi$:= the agent has a *basic (piece of) evidence for* φ.

$B\varphi$:= the agent *believes* φ.

$[\forall]\varphi$:= the agent *infallibly knows* φ (i.e., φ is true in all possible worlds).

The Logic of Evidence, Belief and Infallible Knowledge

Semantics of [van Benthem and Pacuit, 2011]

Given an evidence model $\mathcal{M} = (X, \mathcal{E}_0, V)$ and $x \in X$:

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\mathcal{M}, x \models p & \text{iff} & x \in V(p) \\
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where $[\![\varphi]\!]^{\mathcal{M}} := \{x \in X \mid \mathcal{M}, x \models \varphi\}$.

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where $[\![\varphi]\!]^{\mathcal{M}} := \{x \in X \,|\, \mathcal{M}, x \models \varphi\}$.

So a proposition is believed (in the sense of van Benthem and Pacuit) iff it is supported by all the strongest bodies of evidence, or equivalently iff it is true in all the most plausible worlds.

Example 1

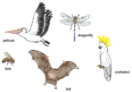
• Alice (a), a biology student, investigates an animal (unknown to her). She receives "pieces of evidence" from 4 different sources of information (her colleagues):

Source 1: it can swim (e_1)

Source 2: non-flying bird (e_2)

Source 3: it lays eggs (e_3)

Source 4: it flies (e_4)



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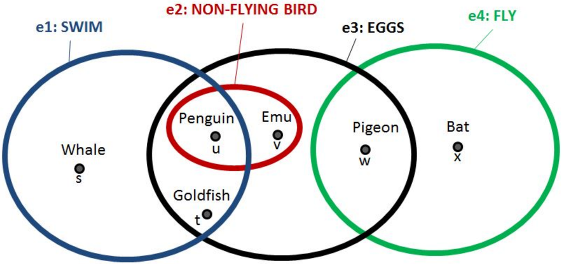

- **Worlds** $X = \{Whale, Penguin, Emu, Goldfish, Pigeon, Bat\}$
- **Evidence pieces** $\mathcal{E} = \{e_1, e_2, e_3, e_4, X\}$

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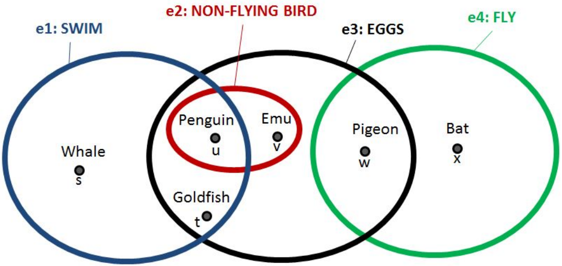

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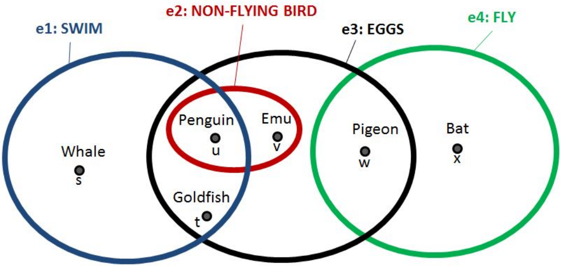

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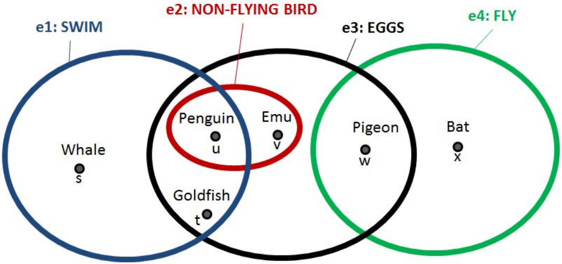

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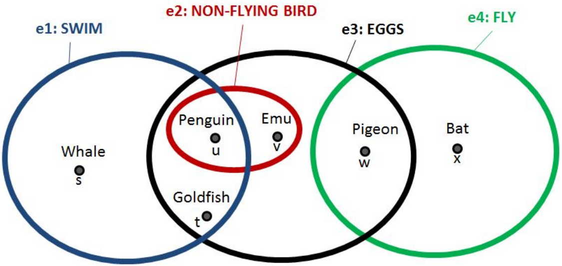

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- **Beliefs**: $B(Penguin \vee Pigeon)$, $B(EGGS)$ (i.e. Be_3).
- **Non-beliefs**: $\neg B(e_1)$, $\neg B(e_2)$, $\neg B(e_4)$.

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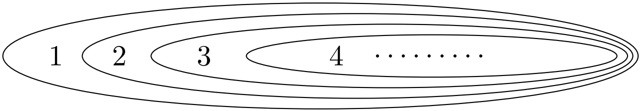
But shouldn't an agent's rational beliefs still be **consistent**?

- when $\mathcal{E}_0$ is finite, beliefs are consistent ($\neg B\bot$).
- BUT: $B\bot$ *can* hold in some "bad" infinite models.

Example 2

$\mathcal{M} = (\mathbb{N}, \mathcal{E}_0, V)$ with $\mathcal{E}_0 = \{[n, \infty) \cap \mathbb{N} \mid n \in \mathbb{N}\}$ and $V(p) = \emptyset$.



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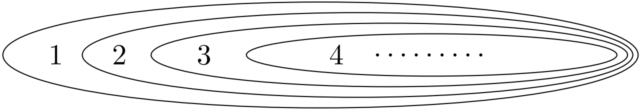

- $\mathcal{E}_0 \in \mathcal{F}$, therefore, $Max_{\subseteq}\mathcal{F} = \{\mathcal{E}_0\}$;
- $\bigcap \mathcal{E}_0 = \emptyset$ implies that $B\bot$ holds in $\mathcal{M}$.

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So we should instead focus on **finite** bodies of evidence: these are guaranteed to be *always consistent*.

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So we should instead focus on **finite** bodies of evidence: these are guaranteed to be *always consistent*.

But of course, in infinite models, there might **not exist any maximal finite body of evidence**.

So instead of focusing on *all the "strongest" such bodies*, we may weaken the definition by looking at *all finite bodies of evidence that are "strong enough"*.

Evidence-Based Belief

Definition

P is *believed* iff it is supported by all *sufficiently strong finite bodies* of evidence.

i.e. *every finite body of evidence can be strengthened to a finite body supporting* P:

$$\forall F \in \mathcal{F}^{fin} \, \exists F' \in \mathcal{F}^{fin} (F \subseteq F' \wedge \bigcap F' \subseteq P)$$

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P is *believed* iff it is supported by all *sufficiently strong finite bodies* of evidence.

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$$\forall F \in \mathcal{F}^{fin} \, \exists F' \in \mathcal{F}^{fin} (F \subseteq F' \wedge \bigcap F' \subseteq P)$$

Unlike the concept of belief of van Benthem & Pacuit, our definition gives us an inherently topological notion.

Evidential Topology

Recall: The family of (combined) evidence $\mathcal{E}$ forms a *topological basis*.

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A *topo-e-model* is a tuple $\mathfrak{M} = (X, \mathcal{E}_0, \tau, V)$, where

- $(X, \mathcal{E}_0, V)$ is an evidence model,
- $\tau = \tau_{\mathcal{E}}$ is the *evidential topology*.

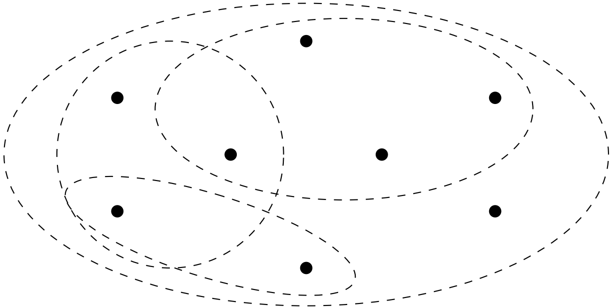
Topological Evidence Models

$$(X, \mathcal{E}_0, \tau_{\mathcal{E}})$$

$$\mathcal{E}_0 \xrightarrow[\text{finite intersections}]{} \mathcal{E} \xrightarrow[\text{arbitrary unions}]{} \tau_{\mathcal{E}}$$

$$\begin{array}{ccc} \Cup & \Cup & \Cup \\ e_0 & e & U \end{array}$$

direct evidence — *combined evidence* — *argument/verifiable prop.*



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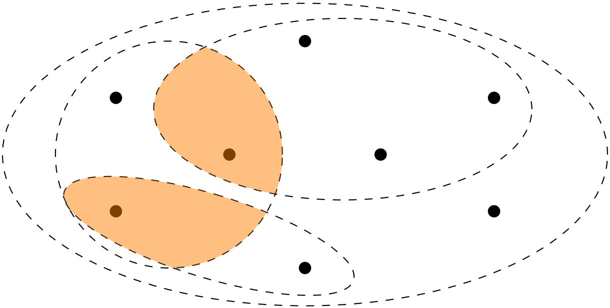

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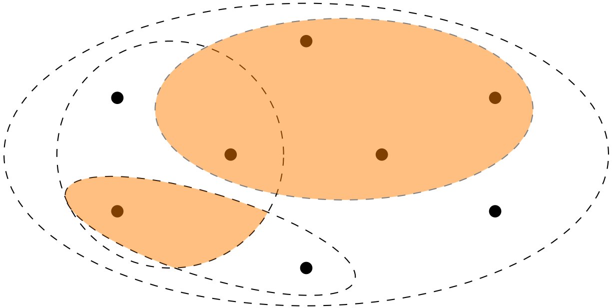
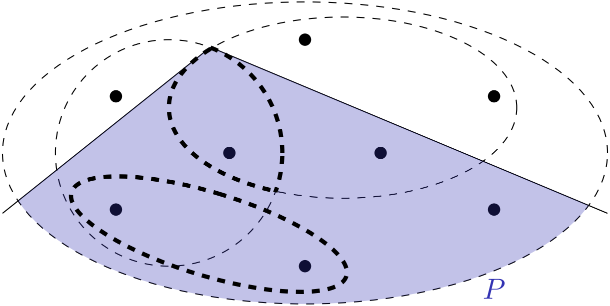


Evidential Support on topo-e-models



e_0 is *basic evidence for* P iff $e_0 \subseteq P$

e is *(combined) evidence for* P iff $e \subseteq P$

U **is an argument for** P **iff** $U \subseteq P$

Argument

An *argument* for P is a *disjunction* $U = \bigcup_{i \in I} e_i$ *of evidences* $e_i \in \mathcal{E}$, *each separately supporting* P (i.e. $e_i \subseteq P$ for all $i \in I$).

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Arguments $\sim$ Open sets

Intermezzo: two more topological notions

Given a topological space (X, τ) and $A \subseteq X$:

A is *dense* if every non-empty open set $U \in \tau$ intersects A, i.e., if for all $U \in \tau \setminus \{\emptyset\}$, $U \cap A \neq \emptyset$. That is, A is dense in (X, τ) iff $Cl(A) = X$.

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A is *nowhere dense* if $Int(Cl(A)) = \emptyset$. Equivalently: if the interior of its complement $Int(X \setminus A)$ is dense (i.e., $Cl(Int(X \setminus A)) = X$).

Properties of nowhere sense sets (recall: $Int(Cl(A)) = \emptyset$)

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These properties make "nowhere dense sets" into a good model for "vanishingly small", or *negligible*.

From nowhere dense to "almost all"

In some papers, "almost all" is taken to mean "all points of the space except for a nowhere dense set" [Baltag et al., 2016, Bjorndahl and Özgün, 2020, Baltag et al., 2022].

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E.g. a property P is said to be true "almost everywhere" if the set of points not satisfying P is nowhere dense.

Note: In other literature, "almost all" means "all except for a meagre set (-countable union of nowhere dense sets). In Probability Theory, "almost all" means "probability 1", i.e. "all except for a set of measure 0".

Justification

A *justification* for P is an *argument* U *for* P *that is consistent with every available evidence* (i.e. $U \in \tau_{\mathcal{E}}$ such that $U \subseteq P$ and $U \cap e \neq \emptyset$ for all $e \in \mathcal{E}$).

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Justifications $\sim$ Dense open sets

An argument (or justification) U is correct at x iff $x \in U$.

Characterizations of Belief

P is believed iff it is entailed by all "sufficiently strong" evidence.

Given a topo-e-model $\mathcal{M} = (X, \mathcal{E}_0, \tau, V)$,

$$
\begin{aligned}
B\varphi \text{ holds } & \text{iff } \forall F \in \mathcal{F}^{fin}\ \exists F' \in \mathcal{F}^{fin}(F \subseteq F' \text{ and } \bigcap F' \subseteq [\![\varphi]\!]) \\
& \text{iff } \forall e \in \mathcal{E}\ \exists e' \in \mathcal{E}(e' \subseteq e \cap [\![\varphi]\!]) \\
& \text{iff } \forall U \in \tau \setminus \{\emptyset\}\ \exists U' \in \tau \setminus \{\emptyset\}(U' \subseteq U \cap [\![\varphi]\!]) \\
& \text{iff } \exists U \in \tau(U \subseteq [\![\varphi]\!] \text{ and } Cl(U) = X) \\
& \text{iff } Int([\![\varphi]\!]) \text{ is dense (i.e. } Cl(Int([\![\varphi]\!])) = X) \\
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\end{aligned}
$$

Exercise: Prove the above equivalences (see [Özgün, 2017, Chapter 5] for the proof).

Characterizations of Belief

To recap, the following are equivalent:

- *P is believed* (i.e., every finite body of evidence can be strengthened to a finite body supporting P);
- *every (combined) evidence can be strengthened to some evidence supporting P*:
  $(\forall e \in \mathcal{E}\ \exists e' \in \mathcal{E}(e' \subseteq e \cap P))$.
- *there exists a justification for P*:
  $\exists U \in \tau(U \subseteq P$ and $Cl(U) = X)$.
- *P includes a dense open set*;
- *$Int(P)$ is dense*:
  $Cl(Int(P)) = X$.
- *$X \setminus P$ is nowhere-dense*:
  $Int(Cl(X \setminus P)) = \emptyset$.

Characterizations of Belief

$B\varphi$ holds iff $Cl(Int(\llbracket\varphi\rrbracket)) = X$

iff $Int(Cl(\llbracket\neg\varphi\rrbracket)) = \emptyset$

iff $\llbracket\neg\varphi\rrbracket$ is *nowhere dense*

iff φ is true in "*almost all*" epistemically possible states

Characterizations of Belief

$B\varphi$ holds iff $Cl(Int(\llbracket\varphi\rrbracket)) = X$
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iff φ is true in "*almost all*" epistemically possible states

Exercise: Prove the above equivalences.

- $B\bot$ never holds, since $Cl(Int(\emptyset)) = \emptyset$.
- The logic of belief is KD45_B (wrt the class of all topo-e-models).

Rational Belief is Justified Belief

So our definition really gives us a concept of justified belief: belief for which there exists some evidential justification.

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Rational Belief is Justified Belief

So our definition really gives us a concept of justified belief: belief for which there exists some evidential justification.

When $\mathcal{E}_0$ is finite, our definition is equivalent to the one of van Benthem & Pacuit (2011).

But in general, our notion is better behaved.

Topologically natural:
P is believed iff it's true in "almost all" worlds: i.e. *all except for a nowhere-dense set*.

Logically well-behaved:
Our notion of belief is always consistent (i.e. $B\bot$ never holds, since $Cl(Int(\emptyset)) = \emptyset$).

Overview Table

EPISTEMOLOGY	TOPOLOGY
Basic Evidence	Subbasis ($\mathcal{E}_0$)
(Combined) Evidence	Basis ($\mathcal{E}$)
Arguments	Open Sets ($\tau_{\mathcal{E}_0}$)
Justifications	Dense Open Sets
Justified Belief	Dense Interior (nowhere-dense complement)

A realistic notion of "Fallible" (Defeasible) Knowledge

Infallible knowledge $\forall$ is too much to ask: it requires absolute certainty.

In that sense, we "know" very few things (maybe only logical-mathematical tautologies, or maybe also things known by introspection: "I exist" etc.)

Epistemologists proposed various "softer" notions, representing types of "fallible knowledge" (not absolutely certain).

We define **(fallible) knowledge K as "correctly justified belief"**:

P **is known in world** x iff **the agent has a correct justification for** P **at** x**.**

Characterizations of Knowledge

NOTE: **Knowledge $\neq$ JTB**!
(cf Gettier counterexamples)
Instead: **Knowledge = belief based on a true justification**!

The following are equivalent:

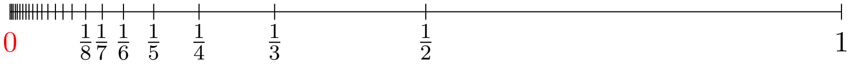
- P **is "fallibly known" at** x (i.e. the agent has a correct justification for P);
- P **includes a dense open neighborhood of** x: $\exists U \in \tau (x \in U \subseteq P$ and $Cl(U) = X)$.
- **Interior of** P **is dense and contains the actual world** x: $x \in Int(P)$ and $Cl(Int(P)) = X$.

Overview Table

EPISTEMOLOGY	TOPOLOGY
Basic Evidence	Family of generators of a topology (E_0)
(Combined) Evidence	(Topological) Basis (E)
Arguments	Open Sets (τ_E)
Justifications	Dense Open Sets
Belief	Dense interior (nowhere-dense complement)
There exists factive evidence (argument) for P	$x \in Int(P)$
Knowledge = There exists a factive justification for P	$x \in Int(P)$ which is Dense

Example 3: Knowledge $\neq$ JTB

$\mathcal{M} = ([0,1], \mathcal{E}_0, \tau, V)$ with $\mathcal{E}_0 = \{(a,b) \cap [0,1] \mid a,b \in \mathbb{R}, a < b\}$



$P = [0,1] \setminus \{\frac{1}{n} : n \in \mathbb{N}\}$ and $\neg P = \{\frac{1}{n} : n \in \mathbb{N}\}$

- $Int(P) = P \setminus \{0\}$ and $Cl(Int(P)) = [0,1]$
- BP holds (everywhere)
- KP holds at every state in P, *except at* 0:

$$0 \notin Int(P) = P \setminus \{0\}$$

- $0 \models BP \wedge P$, but $0 \not\models KP$ (no true justification for P at 0).

A Gettier-Type Counterexample - Blocked!

Suppose that I have strong evidence for the proposition:

(a) Sophia owns a Ford.

My evidence might be that Sophia has at all times in the past, as far as I remember, owned a car, and always a Ford, and that she has just offered me a ride while driving a Ford. (Unbeknownst to me, it was in fact a rental car.)

I have another friend, Fernando, and I had no idea about where Fernando was last week. On the basis of (a), I believe that

(b) Sophia owns a Ford or Fernando was in Beijing last week.

I am thereby justified in believing (b). As it turns out, unbeknownst to me, Fernando was indeed in Beijing last week. Therefore, my justified belief in (b) is true.

Connection with the interior semantics in extremely disconnected spaces

Note that in this refined setting, the interior operator **NO LONGER** represents "knowledge", but only "having factual evidence for".

Still, is there a connection to the interior semantics for knowledge, and to the semantics of Stalnaker's full belief in extremally disconnected spaces?

Connection with the interior semantics in extremely disconnected spaces

Yes!

Connection with the interior semantics in extremely disconnected spaces

Yes!

Recall that every topology has an extremely disconnected topology inside:

According to Homework 2, the dense open sets of any given topology τ form an extremally disconnected topology τ_{dense}, where $\tau_{dense} = \{U \in \tau : Cl(U) = X\} \cup \{\emptyset\}$.

Exercise

Check that our new definitions of justified belief B and (fallible) knowledge K in this lecture are equivalent to putting

$$KP = Int_{dense}(P),$$

$$BP = Cl_{dense}(Int_{dense}(P)).$$

The logic of evidence, knowledge, and belief

$$\mathcal{L}^+ := p \mid \neg\varphi \mid \varphi \wedge \varphi \mid E_0\varphi \mid \Box_0\varphi \mid E\varphi \mid \Box\varphi \mid B\varphi \mid K\varphi \mid \forall\varphi$$

$E_0\varphi$:= the agent has a *basic (piece of) evidence* supporting φ.

$\Box_0\varphi$:= the agent has a *factive piece of evidence* for φ.

$E\varphi$:= the agent has *(combined) evidence* for φ.

$\Box\varphi$:= the agent has *factive (combined) evidence* for φ.

$B\varphi$:= the agent has a justified belief in φ.

$K\varphi$:= the agent knows φ (in the fallible sense).

$\forall\varphi$:= the agent infallibly knows φ.

The logic of evidence, knowledge, and belief

Given a topo-e-model $\mathcal{M} = (X, \mathcal{E}_0, \tau, V)$ and $x \in X$, we interpret $\mathcal{L}^+$ recursively as follows:

$$
\begin{array}{lll}
\mathcal{M}, x \models E_0\varphi & \text{iff} & (\exists e \in \mathcal{E}_0)(e \subseteq [\![\varphi]\!]^{\mathcal{M}}) \\
\mathcal{M}, x \models \Box_0\varphi & \text{iff} & (\exists e \in \mathcal{E}_0)(x \in e \subseteq [\![\varphi]\!]^{\mathcal{M}}) \\
\mathcal{M}, x \models E\varphi & \text{iff} & (\exists e \in \mathcal{E})(e \subseteq [\![\varphi]\!]^{\mathcal{M}}) \\
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\mathcal{M}, x \models B\varphi & \text{iff} & Cl(Int([\![\varphi]\!]^{\mathcal{M}})) = X \\
\mathcal{M}, x \models K\varphi & \text{iff} & x \in Int([\![\varphi]\!]^{\mathcal{M}}) \text{ and } Cl(Int([\![\varphi]\!]^{\mathcal{M}})) = X \\
\mathcal{M}, x \models \forall\varphi & \text{iff} & [\![\varphi]\!] = X
\end{array}
$$

Observation: $[\![\Box\varphi]\!]^{\mathcal{M}} = Int[\![\varphi]\!]^{\mathcal{M}}$
(The interior-based topological semantics)

★ *We might have a few too many modal operators here!*

The logic of (factive) evidence

$$\mathcal{L} := p \mid \neg\varphi \mid \varphi \wedge \varphi \mid \Box_0\varphi \mid \Box\varphi \mid \forall\varphi$$

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All other modalities are definable in this language:

$$\begin{aligned} E_0\varphi &:= \exists\Box_0\varphi & \qquad B\varphi &:= \forall\Diamond\Box\varphi \\ E\varphi &:= \exists\Box\varphi & \qquad K\varphi &:= \Box\varphi \wedge B\varphi \end{aligned}$$

where

$$\exists\varphi := \neg\forall\neg\varphi.$$

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where

$$\exists\varphi := \neg\forall\neg\varphi.$$

Exercise: Prove the above equivalences.

Axiomatization

the S5 axioms and rules for $\forall$

the S4 axioms and rules for $\Box$

$\Box_0\varphi \to \Box_0\Box_0\varphi$

$\forall\varphi \to \Box_0\varphi$

$\Box_0\varphi \to \Box\varphi$

$(\Box_0\varphi \wedge \forall\psi) \to \Box_0(\varphi \wedge \forall\psi)$

from $\varphi \to \psi$, infer $\Box_0\varphi \to \Box_0\psi$

Theorem

The logic of evidence has the finite model property, is decidable, and is completely axiomatized by the above system.

Fragments

The sound and complete logic of belief (=the B fragment of our logic) is axiomatized by the system $\mathsf{KD45}_B$.

The sound and complete logic of (fallible) knowledge (=the K fragment) is axiomatized by the system $\mathsf{S4.2}_K$.

The sound and complete logic of knowledge and belief (=the KB fragment) is completely axiomatized by Stalnaker's axioms for doxastic-epistemic logic Stal.

Further extensions

- Multi-agent extensions

  dos Santos Gomes, D. (2025) *Virtual Group Knowledge on Topological Evidence Models*. Master's thesis, ILLC, University of Amsterdam.

  Ramirez, A. I. R. (2015) *Topological models for group knowledge and belief.* Master's thesis, ILLC, University of Amsterdam.

- Multi-agent + Completeness results with respect to specific topological spaces

  Baltag, A., Bezhanishvili, N. and Fernańdez González, S. (2019) *The McKinsey-Tarski Theorem for Topological Evidence Logics*. Proceedings of WoLLIC 2019: 177-194.

  Fernańdez González, S. (2018) *Generic Models for Topological Evidence Logics.* Master's thesis, ILLC, University of Amsterdam.

- Evidence diffusion in Social Networks

  Zotescu, T.Ş (2024) *Multi-agent Topological Models for Evidence Diffusion*. Master's thesis, ILLC, University of Amsterdam.

  Özgün, A., Smets, S., Zotescu, T.Ş (2015) *Evidence Diffusion in Social Networks: a Topological Perspective*. Proceedings of LORI 2025. Forthcoming.

Further extensions

- The relational definition of belief in terms of evidential plausibility order and alternative relational settings

  Baltag, A. and Liberman, A. O. (2017) *Evidence Logics with Relational Evidence*. Proceedings of LORI 2017: 17-32.

  Fiutek, V. (2013). *Playing with Knowledge and Belief*. PhD thesis. University of Amsterdam.

  Liberman, A. O. (2016). *Dynamic Evidence Logics with Relational Evidence.* Master's thesis, ILLC, University of Amsterdam.

- Dynamics of evidence management

  van Benthem, J. and Pacuit, E. *Dynamic Logics of Evidence-Based Beliefs*. Studia Logica (2011) 99: 61.

  Özgün, A. (2017) *Evidence in Epistemic Logic: A topological perspective*. PhD thesis. Université de Lorraine & University of Amsterdam - Chapter 5.

Further extensions

- Quantitative extensions

  Pinto Prieto, D., de Haan R., and Özgün, A. , *A Belief Model for Conflicting and Uncertain Evidence: Connecting Dempster-Shafer Theory and the Topology of Evidence*. Proceedings of KR 2023.

  Fiutek, V. (2013). *Playing with Knowledge and Belief*. PhD thesis. University of Amsterdam.

  Baltag, A., Fiutek, V., and Smets., S. (2016). *Belief and Evidence in Justification Models*. In Advances in Modal Logic, vol. 11, pp. 156-176, (Eds) Lev Beklemishev, Stéphane Demri and András Máté, College Publications.

Extra, time permitting: Defeasible Knowledge

Interaction Between Knowledge and Belief

Platonic equation:

knowledge = justified true belief (JTB) $\quad + \quad$ (??)

"an agent knows φ iff φ is true, they believe that it is true and they are justified in believing that φ."

There are many proposals for (??). It is not an easy task, if possible, to identify a unique notion of knowledge that can deal with all kinds of intuitive counterexamples.

One can accept that all these proposals "*capture important intuitions that can in some way or other be regarded as relevant to the question whether or not a given belief constitutes a piece of knowledge*" (Rott, 2004, p. 469).

Interaction Between Knowledge and Belief

Platonic equation:

knowledge = justified true belief (JTB) $+$ (??)

"an agent knows φ iff φ is true, they believe that it is true and they are justified in believing that φ."

We can now talk about the defeasibility and stability analysis of knowledge (Lehrer and Paxson, 1969; Lehrer, 1990; Klein, 1971, 1981; Stalnaker, 2006).

Our knowledge is *not stable*

- *irrevocable knowledge*: cannot be defeated by any evidence gathered later
- *stable knowledge*: cannot be defeated by any factive evidence gathered later

Stability theory of knowledge
an agent knows P :

1. P is true
2. she believes that P is true
3. her **belief** in P cannot be defeated by new *factive* information. *stable belief*

Our knowledge is *defeasible*

- *irrevocable knowledge*: cannot be defeated by any evidence gathered later
- *stable knowledge*: cannot be defeated by any factive evidence gathered later

In-defeasibility theory of knowledge
an agent *"indefeasibly"* knows P :

1. P is true
2. she believes that P is true
3. her **belief** in P cannot be defeated by new *factive* information. *stable belief*
4. her **justification** is undefeated by new *factive* information. *stable justification*

Our knowledge is *defeasible*

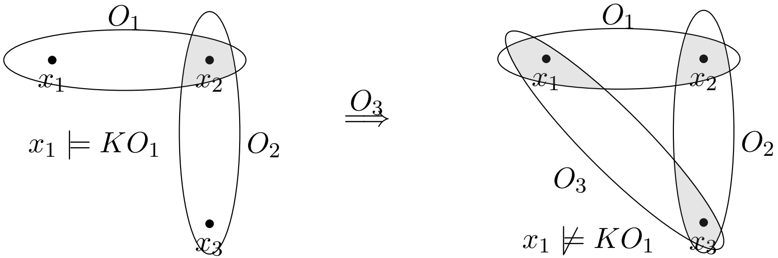
BP holds (everywhere) iff $Cl(Int(P)) = X$

KP holds at x iff $x \in Int(P)$ and $Cl(Int(P)) = X$

Our knowledge is *defeasible*

BP holds (everywhere) iff $Cl(Int(P)) = X$

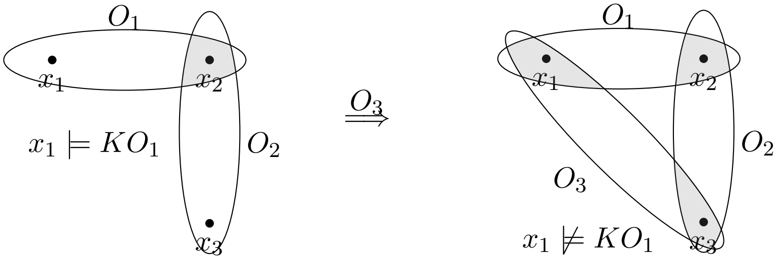
KP holds at x iff $x \in Int(P)$ and $Cl(Int(P)) = X$



Our knowledge is *defeasible*

$$BP \text{ holds (everywhere) iff } Cl(Int(P)) = X$$

$$KP \text{ holds at x iff } x \in Int(P) \text{ and } Cl(Int(P)) = X$$



O_3 is a *misleading defeater*: $O_2 \cap O_3 = \{x_3\} \Rightarrow$ false evidence.

Non-misleading defeaters

K is defeasible for factive evidence, but *in-defeasible* for "**non-misleading**" evidence.

Given a topo-e-model $\mathcal{M} = (X, \mathcal{E}_0, \tau, V)$ and $x \in X$,

$Q \subseteq X$ is *misleading* iff it is *new* and it produces some false new evidence.

$Q \subseteq X$ is *misleading* iff $x \notin Q \cap e \notin \mathcal{E} \cup \{\emptyset\}$ for some $e \in \mathcal{E}$.

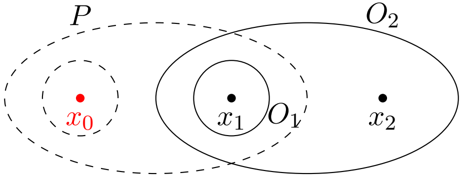
Topologically, misleading evidence adds an open set to the evidential topology that does not include the actual state.

The Weak Stability and Defeasibility

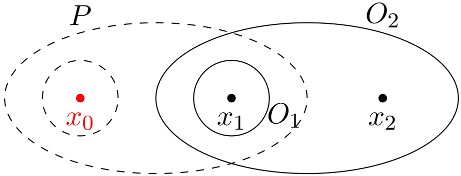
1. P is true
2. she believes that P is true
3. her **belief** in P cannot be defeated by new *non-misleading* evidence. *weak stable belief*

$$\not\Downarrow$$
$$knowledge$$

Our knowledge is *weakly stable*

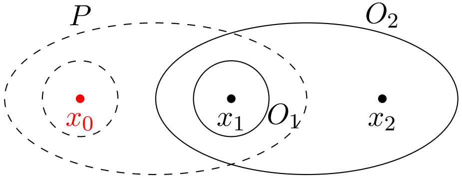


Our knowledge is *weakly stable*



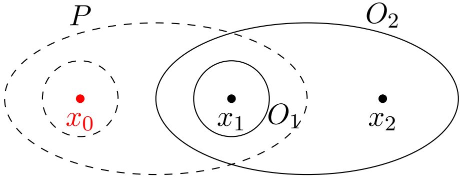
- P is true (at x_0)
- BP holds, since $Cl(Int(P)) = Cl\{x_1\} = X$

Our knowledge is *weakly stable*



- P is true (at x_0)
- BP holds, since $Cl(Int(P)) = Cl\{x_1\} = X$
- BP is weakly stable

Our knowledge is *weakly stable*



- P is true (at x_0)
- BP holds, since $Cl(Int(P)) = Cl\{x_1\} = X$
- BP is weakly stable
- $x_0 \not\models KP$, since $x_0 \notin Int(P) = \{x_1\}$

The Weak Stability and Defeasibility

an agent knows P :

1. P is true
2. she believes that P is true
3. her **belief** in P cannot be defeated by new *non-misleading* evidence. *weak stable belief*
4. the **belief in its justification** is undefeated by new *non-misleading* evidence. *weak stable justification*

Our knowledge is *weakly in-defeasible*

an agent knows P :

1. P is true
2. she believes that P is true
3. her **belief** in P cannot be defeated by new *non-misleading* evidence. *weak stable belief*
4. the **belief in its justification** is undefeated by new *non-misleading* evidence. *weak stable jutification*

$x \models KP$ iff $\exists U \in \tau \setminus \{\emptyset\}$ s.t. $U \subseteq P$ and $U \cap Q \neq \emptyset$ for all non-misleading Q

Questions?

Baltag, A., Bezhanishvili, N., Özgün, A., and Smets, S. (2016).
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