Topological Approaches to Epistemic Logic

Lecture 4: The Topology of Potential Evidence
-
Subset Space Semantics and TopoLogic

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Motivation

So far we looked at topological evidence models, modelling what the agent comes to know and believe based on the evidence *they have gathered*.

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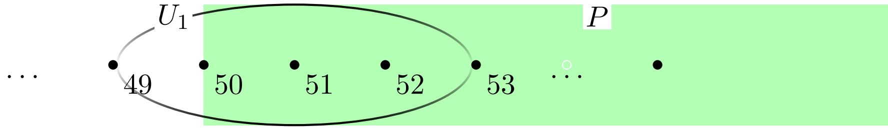
So far we looked at topological evidence models, modelling what the agent comes to know and believe based on the evidence *they have gathered*.

Today, we will consider models where the topology represents all the evidence *the agent can potentially obtain*, and we explicitly refer to the evidence the agent currently possesses.

Example: speeding of a car

A policeman uses a radar with accuracy ± 2 km/h to determine whether a car is speeding in a 50 km/h speed-limit zone. Suppose the radar shows 51 km/h:

$P = (50, \infty) :=$ the car is speeding
$U_1 = (49, 53) :=$ the reading of the 1st-radar is 51 km/h

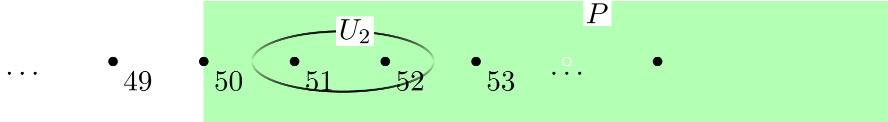


With the measurement $(49, 53)$ in hand, the policeman cannot be said to *know* that the car is speeding: $(49, 53) \not\subseteq S = (50, \infty)$.

Example: speeding of a car

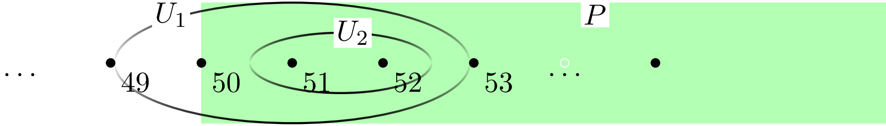
Suppose that the policeman takes another measurement, i.e., spends more *effort*, using a more accurate radar with an accuracy of ± 1 km/h which shows 51.5 km/h.

$P = (50, \infty) :=$ the car is speeding
$U_2 = (50.5, 52.5) :=$ the reading of the 2nd-radar is 51.5 km/h



With the measurement $(50.5, 52.5)$ in hand, the policeman *comes to know* that the car is speeding: $(50.5, 52.5) \subseteq S = (50, \infty)$.

Example: speeding of a car



- $X = (0, \infty)$ as *the set of possible worlds*, where we assume the car is known to be *moving*;
- $\mathcal{B} = \{(a, b) \in \mathbb{Q} \times \mathbb{Q} : 0 < a < b < \infty\}$ as *possible measurement results by arbitrarily accurate radars*.
- $\mathcal{B}$ is a topological basis over X, and the topology τ generated by $\mathcal{B}$ is the *standard topology on real numbers* (restricted to X).

Contents

Subset Space Semantics and Topo-Logic

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Subset Space Semantics and Topo-Logic

Topologic was introduced in [Moss and Parikh, 1992], to capture the relationship between *effort* and *knowledge*.

It is a *single agent* logic with two modalities:

- $K\varphi$: the agent knows φ.
- $\blacksquare\varphi$ for (evidence-gathering) *effort*:

$$\blacksquare\varphi : \text{ after any effort, } \varphi \text{ is still true.}$$

Note: $\blacksquare$ is not the normal modal operator we know from modal logic.

Effort could be *measurement, computation, approximation, learning, hearing announcement*, etc.

Subset Space Semantics (SSL)

$$(\mathcal{L}_{K\blacksquare}) \quad \varphi ::= p \mid \neg\varphi \mid \varphi \wedge \varphi \mid K\varphi \mid \blacksquare\varphi$$

$K\varphi :=$ the agent *infallibly knows* φ

$\blacksquare\varphi :=$ φ is *stably true (under any further evidence-gathering)*

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Definition 1 (Subset Space Model)

A *subset space* is a pair $(X, \mathcal{O})$ where X is a non-empty set and $\mathcal{O} \subseteq \mathcal{P}(X)$. A *subset space model* is a tuple $(X, \mathcal{O}, V)$ where $V : \mathrm{Prop} \to \mathcal{P}(X)$.

Note: $(X, \mathcal{O})$ is not necessarily a topological space.

Formulas of $\mathcal{L}_{K\blacksquare}$ are interpreted with respect to pairs of the form (x, U) with $x \in U \in \mathcal{O}$, called *epistemic scenarios*.

- x represents the actual state, and U represents *the agent's current evidence, e.g., result of her measurement*.

Subset Space Semantics (SSL)

$$(\mathcal{L}_{K\blacksquare}) \quad \varphi ::= p \mid \neg\varphi \mid \varphi \wedge \varphi \mid K\varphi \mid \blacksquare\varphi$$

Given a subset space model $\mathcal{X} = (X, \mathcal{O}, V)$ and an epistemic scenario (x, U) of $\mathcal{X}$:

$\mathcal{X}, (x, U) \models p$	iff	$x \in V(p)$
$\mathcal{X}, (x, U) \models \neg\varphi$	iff	$\mathcal{X}, (x, U) \not\models \varphi$
$\mathcal{X}, (x, U) \models \varphi \wedge \psi$	iff	$\mathcal{X}, (x, U) \models \varphi$ and $\mathcal{X}, (x, U) \models \psi$
$\mathcal{X}, (x, U) \models K\varphi$	iff	$(\forall y \in U)(\mathcal{X}, (y, U) \models \varphi)$
$\mathcal{X}, (x, U) \models \blacksquare\varphi$	iff	$\forall O \in \mathcal{O}(x \in O \subseteq U \Rightarrow \mathcal{X}, (x, O) \models \varphi)$

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$\mathcal{X}, (x, U) \models \blacksquare\varphi$	iff	$\forall O \in \mathcal{O}(x \in O \subseteq U \Rightarrow \mathcal{X}, (x, O) \models \varphi)$

Note: Evaluation points, *epistemic scenarios*, are pairs of a point and a set, rather than single points, as in the interior semantics discussed earlier.

SSL: *effort* modality and knowledge

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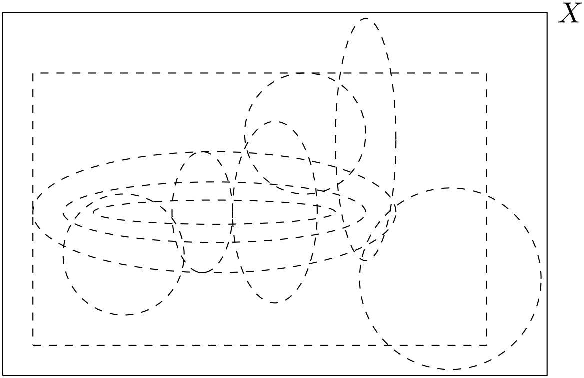

$(X, \mathcal{O})$

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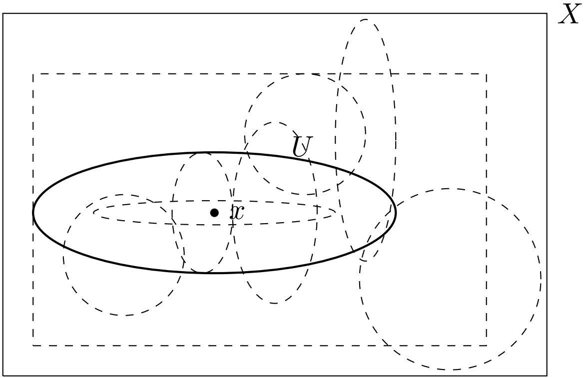

$x :=$ the actual state and $U :=$ the agent's current evidence

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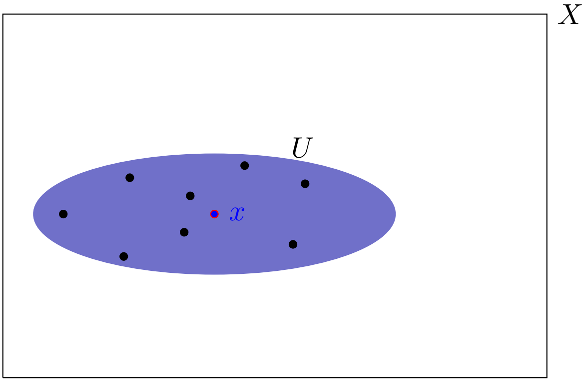

$$(x, U) \models K\varphi \text{ iff } (\forall y \in U)((y, U) \models \varphi)$$

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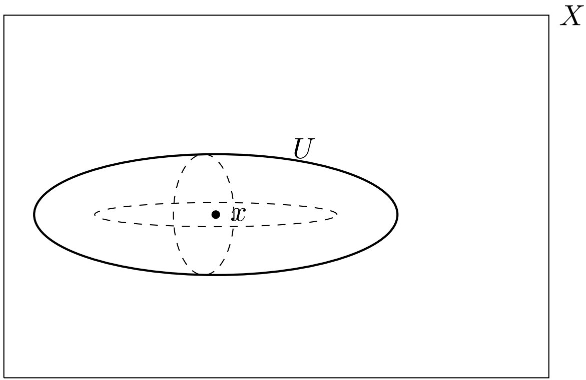

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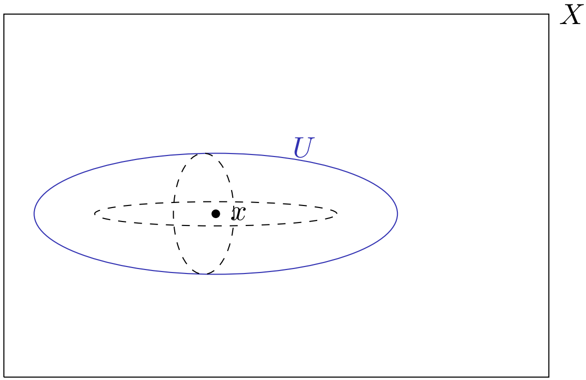

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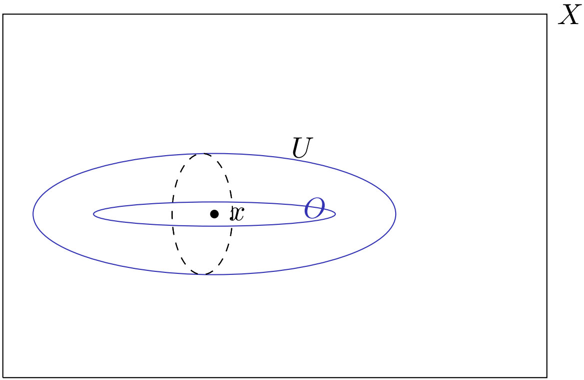

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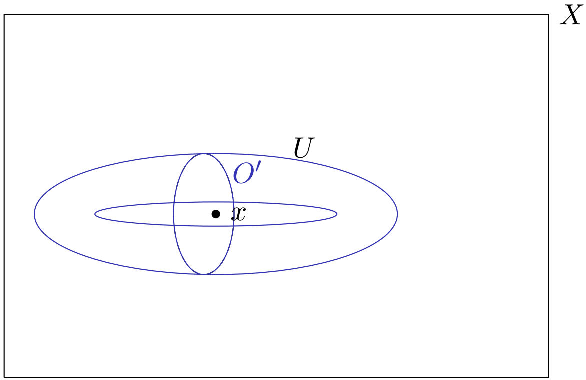
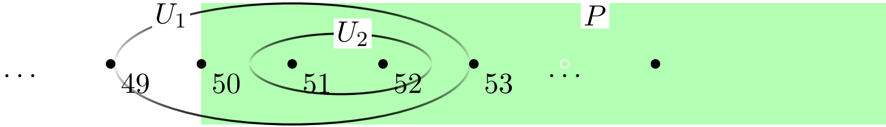


$(x, U) \models \blacksquare\varphi$ iff $\forall O \in \tau(x \in O \subseteq U \Rightarrow (x, O) \models \varphi)$

More on the *effort modality* $\blacksquare\varphi$

- subset space style semantics is rich enough to *distinguish potential evidence from the agent's current evidence*;
- *knowledge* is entailed by the agent's current evidence;
- *more effort* corresponds to a smaller neighbourhood, to a better approximation of where the real world is.

Speeding $P = (50, \infty)$ is verifiable with certainty

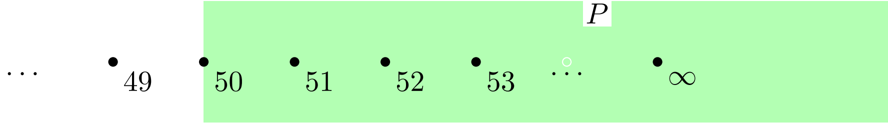


- The policeman doesn't know P with certainty in U_1.
- But *P is verifiable with certainty*. He can always get a better measurement in which P is infallibly known!

$$(x, U_1) \models p \to \blacklozenge Kp$$

- For instance in U_2, P is infallibly known.

Not speeding $X \setminus P = (-\infty, 50]$ is not verifiable with certainty

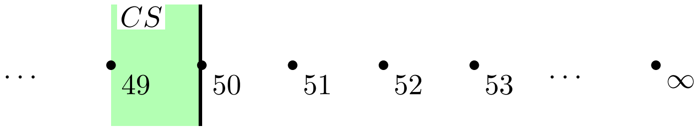


If the speed of the car is exactly 50 km/h, then the car is not speeding, but the policeman will never know that!

So $X \setminus P$ is not always verifiable with certainty, hence P itself is not always falsifiable.

Nevertheless, $X \setminus P$ is always falsifiable: if false (i.e. if the speed is in P, so that car is speeding), then as we saw the policeman will come to infallibly know that (by some more accurate measurement).

Being closed to speeding $CS = (49, 50]$ is neither verifiable nor falsifiable with certainty



If $x = 50$ then CS is true, but CS will never be known for sure (verified).

If $x = 49$ then CS is false, but CS will never be falsified with certainty.

Validity

- φ is *valid in a model* $\mathcal{X}$, and write $\mathcal{X} \models \varphi$, if $\mathcal{X}, (x, U) \models \varphi$ for all epistemic scenarios (x, U) in $\mathcal{X}$.
- φ is *valid*, denoted $\models \varphi$, if $\mathcal{X} \models \varphi$ for all models $\mathcal{X}$. for all X.
- $[\![\varphi]\!]^U_{\mathcal{X}} = \{x \in U : \mathcal{X}, (x, U) \models \varphi\}$ is the *truth set*, or equivalently, *extension of φ under U in the model $\mathcal{X}$*. We again omit the notation for the model, writing simply $(x, U) \models \varphi$ and $[\![\varphi]\!]^U$, whenever $\mathcal{X}$ is fixed.

These definitions can be given for more restricted classes of models in the standard way.

Axiomatizations

($S5_K$)	the S5 axioms and rules for K
($S4_{\blacksquare}$)	the S4 axioms and rules for $\blacksquare$
(AP)	$(p \to \blacksquare p) \wedge (\neg p \to \blacksquare \neg p)$, for all $p \in \mathrm{Prop}$
(CA)	$K\blacksquare\varphi \to \blacksquare K\varphi$

Table: Logic SSL

Theorem 2 ([Moss and Parikh, 1992])
SSL *is sound and complete with respect to the class of all subset spaces.*

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(CA)	$K\blacksquare\varphi \to \blacksquare K\varphi$
(WD)	$\blacklozenge\blacksquare\varphi \to \blacksquare\blacklozenge\varphi$
(Un)	$\blacklozenge\varphi \wedge \hat{K}\blacklozenge\psi \to \blacklozenge(\blacklozenge\varphi \wedge \hat{K}\blacklozenge\psi \wedge K\blacklozenge\hat{K}(\varphi \vee \psi))$

Table: TopoLogic

Theorem 3 ([Georgatos, 1993, Georgatos, 1994])

TopoLogic *is sound and complete with respect to the class of all topological spaces. Moreover, it has the finite model property, therefore, it is decidable.*

Contents

Interior as *Knowability*

[Bjorndahl, 2018] proposes a topological semantics for *a notion of knowability* in terms of the interior operator.

- He intends to capture a notion of *knowability as potential knowledge*.

φ is knowable := existence of a piece of truthful evidence entailing φ

This notion of knowability can be naturally formalized by the topological notion of the interior of a set:

$$x \in Int(A) \text{ iff } (\exists U \in \tau)(x \in U \subseteq A)$$

Knowledge and Knowability in Subset Space Semantics

$$(\mathcal{L}_{K\Box}) \quad \varphi ::= p \mid \neg\varphi \mid \varphi \wedge \varphi \mid K\varphi \mid \Box\varphi$$

Given a topological model $\mathcal{X} = (X, \tau, V)$ and an epistemic scenario (x, U) of $\mathcal{X}$,

$$\begin{array}{lll} (x,U) \models K\varphi & \text{iff} & (\forall y \in U)((y,U) \models \varphi) \\ (x,U) \models \Box\varphi & \text{iff} & (\exists O \in \tau)(x \in O \subseteq [\![\varphi]\!]^U) \\ & \text{iff} & x \in Int([\![\varphi]\!]^U)^1 \end{array}$$

[1] This $\Box$-operator is *formally* similar to the $\Box$-operator we studied on topo-e-models [Özgün, 2017, Baltag et al., 2022]. Their meanings are different though.

Axiomatization $\mathbf{EL}_{K\Box}$

$$EL_{K\Box} := \mathsf{S5}_K + \mathsf{S4}_\Box + (K\varphi \to \Box\varphi)$$

Theorem 4 (Shehtman, 1999)

$EL_{K\Box}$ is sound and complete w.r.t. all topological spaces. Moreover, it has the finite model property, therefore, it is decidable.[2]

[2] Also see [Goranko and Passy, 1992, Bennett, 1996, Shehtman, 1999, Aiello, 2002].

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Q. Does it make sense to have S5 for K and S4 for $\Box$? In particular, why have Negative Introspection for K but not for $\Box$?

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Q. What about belief?

[2] Also see [Goranko and Passy, 1992, Bennett, 1996, Shehtman, 1999, Aiello, 2002].

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- We argue that the plausibility of the principles Stalnaker proposes relating knowledge and belief relies on a subtle equivocation between:
  (1) an "evidence-in-hand" conception of knowledge, and
  (2) a weaker "evidence-out-there" notion of what *could come to be known*.

Back to Stalnaker's Logic Stal

In [Bjorndahl and Özgün, 2020], we refine and extend Stalnaker's logic of knowledge and belief, Stal.

- We argue that the plausibility of the principles Stalnaker proposes relating knowledge and belief relies on a subtle equivocation between:
  (1) an "evidence-in-hand" conception of knowledge, and
  (2) a weaker "evidence-out-there" notion of what *could come to be known*.
- We import Stalnaker's principles into a "richer" semantic setting based on *topological subset spaces*.
  - These models are rich enough to respect the distinction between (1) and (2), yielding a trimodal logic of knowledge, *knowability*, and belief.

Recall: Stalnaker's System

Stalnaker (2006) has proposed a logic intended to capture the relationship between knowledge and belief, where belief is interpreted in the strong sense of *subjective certainty*.

$$(\mathcal{L}_{KB})\quad \varphi ::= p \mid \neg\varphi \mid \varphi \wedge \psi \mid K\varphi \mid B\varphi$$

This logic extends the classic S4 system for knowledge...

(K_K)	$K(\varphi \to \psi) \to (K\varphi \to K\psi)$	Distribution
(T_K)	$K\varphi \to \varphi$	Factivity
(4_K)	$K\varphi \to KK\varphi$	Positive introspection
(Nec_K)	from φ infer $K\varphi$	Necessitation

Table: $\mathsf{S4}_K$ axioms for knowledge

Recall: Stalnaker's System

...with the following additional axioms.

(D_B)	$B\varphi \to \neg B \neg \varphi$	Consistency of belief
(sPI)	$B\varphi \to KB\varphi$	Strong positive introspection
(sNI)	$\neg B\varphi \to K\neg B\varphi$	Strong negative introspection
(KB)	$K\varphi \to B\varphi$	Knowledge implies belief
(FB)	$B\varphi \to BK\varphi$	Full belief

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Table: Stalnaker's additional axioms

Belief as *subjective certainty*: an agent who feels certain that φ is true also feels certain that she *knows* that φ is true.

Recall - Belief as *the closure of the interior*

In this system, one can prove the following striking equivalence:

$$B\varphi \leftrightarrow \hat{K}K\varphi,$$

where $\hat{K}$ abbreviates $\neg K \neg$.

- Belief is equivalent to "the epistemic possibility of knowledge".
- In particular, belief can be *defined* in terms of knowledge—once you have knowledge, you get belief for free.

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Recall: The interior-based topological semantics

$$\begin{array}{rcl} [\![K\varphi]\!] & = & Int([\![\varphi]\!]) \\ [\![\hat{K}\varphi]\!] & = & Cl([\![\varphi]\!]). \end{array}$$

Then:

$$[\![B\varphi]\!] = Cl(Int([\![\varphi]\!]))$$

Stalnaker's System

Let's have a closer look at Stalnaker's axioms, in particular, focus on (KB) and (FB).

$$K\varphi \to B\varphi$$

"If the agent knows φ, then they believe φ."

- Knowledge is stronger than belief.

$$B\varphi \to BK\varphi$$

"If the agent believes φ, then they believe that they know φ."

- Specific to Stalnaker's notion of belief: *an agent who feels certain that φ is true also feels certain that they know that φ is true.*

Stalnaker's System

Given the strong sense of belief Stalnaker seeks to capture, each of (KB) and (FB) has a certain plausibility.

Tension between (KB) and (FB) emerges when knowledge is interpreted more concretely in terms of what is justified by a body of evidence.

Knowledge from evidence "in hand"

A simple example: Speeding car example!

Another simple example: you've measured your height to be 1.6m, ± 2 cm. With this measurement in hand, you might be said to *know* that you are less than 1.7m tall (having ruled out the possibility that you are taller).

We called this the *evidence-in-hand* conception of knowledge.

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- If you have evidence-in-hand that entails φ, you should be certain of φ.

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This fits well with (KB) ($K\varphi \to B\varphi$).

- If you have evidence-in-hand that entails φ, you should be certain of φ.

It does *not* sit comfortably with (FB) ($B\varphi \to BK\varphi$).

- You can be (subjectively) certain of φ without also being certain that you currently have evidence-in-hand that guarantees φ.

Knowledge from evidence "out there"

Consider now a weaker, *existential* interpretation of "available evidence": *there is* evidence (somewhere out there that the agent *in principle* has) entailing φ.

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- Intuitively, we've shifted from what's *known* to what's *knowable*.

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- Only believe what you think you could come to know.

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- If you are certain of φ, then you are certain that there is evidence entailing φ.
- Only believe what you think you could come to know.

(KB) $(K\varphi \to B\varphi)$ falters.

- The mere fact that you could, in principle, discover evidence entailing φ should not in itself imply that you believe φ.

Knowledge and knowability

It seems we want the "evidence-in-hand" intuition for (KB), and the "evidence-out-there" intuition for (FB).

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> Does “evidence-out-there” intuition (*available evidence*) remind us of anything from previous lectures?

Knowledge and knowability

It seems we want the "evidence-in-hand" intuition for (KB), and the "evidence-out-there" intuition for (FB).

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$$x \in Int(A) \text{ iff } (\exists U \in \tau)(x \in U \text{ implies } U \subseteq A).$$

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- Write $K\varphi$ for "φ is entailed by the evidence-in-hand".
  - Gloss: "φ is known".
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(KB) stays the same.

(FB) becomes (RB), "responsible belief":

$$(B\varphi \to BK\varphi) \rightsquigarrow (B\varphi \to B\Box\varphi).$$

Knowledge and knowability

We now interpret Stalnaker's logic enriched with $\Box$ as *knowability* in topological subset spaces.

Topological Subset Space Semantics

Recall the proposal of [Bjorndahl, 2018]:

Given a topo-model $\mathcal{X} = (X, \tau, \nu)$ and an epistemic scenario (x, U) of $\mathcal{X}$,

$$\begin{array}{lll} \mathcal{X}, (x, U) \models K\varphi & \text{iff} & (\forall y \in U)(\mathcal{X}, (y, U) \models \varphi) \\ \mathcal{X}, (x, U) \models \Box\varphi & \text{iff} & x \in Int(\llbracket \varphi \rrbracket^U) \end{array}$$

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- τ as the set of all possible pieces of evidence, or all possible results of measurements: *evidence-out-there*.
- Given an epistemic scenario (x, U), x represents the actual world and U the agent's current evidence, i.e., *evidence-in-hand*.

Logic for knowledge, knowability, and belief: $\mathcal{L}_{K\Box B}$

What about belief?

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Recall that in Stalnaker's system, belief was reducible to knowledge (via $B\varphi \leftrightarrow \hat{K}K\varphi$), obviating the need for a separate semantic clause for B.

Can we take advantage of this?

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Not directly: adding Stalnaker's axioms (with (FB) replaced by (RB)) to $\mathsf{EL}_{K\Box}$ does *not* produce a logic strong enough to reduce belief to knowledge/knowability.

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- ⋆ This is noteworthy: once we carefully distinguish knowledge from knowability, Stalnaker's postulates no longer imply that belief is reducible.

Logic for knowledge, knowability, and belief: $\mathcal{L}_{K\Box B}$

However, we can strengthen $\mathsf{EL}_{K\Box}$ with additional postulates to obtain such a reduction.

Let $\mathsf{Stal}_{K\Box B}$ denote $\mathsf{EL}_{K\Box}$ together with the following:

(K_B)	$B(\varphi \to \psi) \to (B\varphi \to B\psi)$	Distribution of belief
(sPI)	$B\varphi \to KB\varphi$	Strong pos. introspection
(KB)	$K\varphi \to B\varphi$	Knowledge implies belief
(RB)	$B\varphi \to B\Box\varphi$	Responsible belief
(wF)	$B\varphi \to \Diamond\varphi$	Weak factivity
(CB)	$B(\Box\varphi \vee \Box\neg\Box\varphi)$	Confident belief

Our additional axioms

(K_B), (sPI), and (KB) are theorems of Stalnaker's original system.

(RB) is the translation of (FB) we have already discussed.

Logic for knowledge, knowability, and belief: $\mathcal{L}_{K\Box B}$

Both (wF) and (CB) become theorems of Stalnaker's original system if we "forget" the distinction between $\Box$ and K—that is, replace every $\Box$ with K (and every $\Diamond$ with $\hat{K}$).

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Weak factivity:

$$B\varphi \to \Diamond\varphi$$

"If you are certain of φ, then φ cannot be knowably false."

Confident belief:

$$B(\Box\varphi \vee \Box\neg\Box\varphi)$$

"You believe that φ is either knowable or knowably unknowable."

Logic for knowledge, knowability, and belief: $\mathcal{L}_{K\Box B}$

$\mathsf{Stal}_{K\Box B}$ proves the following equivalence:

$$B\varphi \leftrightarrow K\Diamond\Box\varphi.$$

- Belief is definable from knowledge *and knowability*.

Logic for knowledge, knowability, and belief: $\mathcal{L}_{K\Box B}$

Semantically, this equivalence corresponds to the conception of belief as *dense interior* [Özgün, 2017, Baltag et al., 2022].

$$\begin{array}{rll} (x,U) \models B\varphi & \text{iff} & (x,U) \models K\Diamond\Box\varphi \\ & \text{iff} & U \subseteq Cl(Int(\llbracket\varphi\rrbracket^U)) \\ & \text{iff} & U = Cl(Int(\llbracket\varphi\rrbracket^U)) \\ & \text{iff} & \llbracket\varphi\rrbracket^U \text{ has dense interior in } U. \end{array}$$

- Dense interior is a standard topological notion of largeness.
    - These are precisely the sets with *nowhere dense* complements.

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- Dense interior is a standard topological notion of largeness.
    - These are precisely the sets with *nowhere dense* complements.

Intuitively, such a set fills “almost all” of the space. Morally, then:

$$(x,U) \models B\varphi \quad \text{iff} \quad \text{for “almost all” } y \in U,\ (y,U) \models \varphi.$$

So, while knowledge is interpreted (as usual) as truth in *all* possible alternatives, belief becomes truth in *almost all* possible alternatives.

Logic for knowledge, knowability, and belief: $\mathcal{L}_{K\Box B}$

Theorem 5
$\mathsf{Stal}_{K\Box B}$ *is a sound and complete axiomatization of* $\mathcal{L}_{K\Box B}$ *with respect to the class of topological subset models.*

Theorem 6
$\mathsf{Stal}_{K\Box B}$ *proves all the* KD45 *principles for belief. In fact,* $\mathsf{KD45}_B$ *is a sound and complete axiomatization of the fragment* $\mathcal{L}_B$ *with respect to the class of topological subset models.*

Weaker notions of belief

We adopted weak factivity (wF) and confident belief (CB) in order to obtain a reduction result for belief analogous to Stalnaker's.

Of course, we could drop one or both of these principles.

- In this case, belief is no longer reducible, so we need to augment topological subset models to provide the structure necessary to interpret belief as a primitive.

Weaker notions of belief

Let $\mathsf{EL}_{K\Box B}$ be the logic obtained by dropping the axioms (wF) and (CB) from $\mathsf{Stal}_{K\Box B}$.

As before, we rely on topological subset models; however, we now define the evaluation of formulas with respect to *epistemic-doxastic (e-d) scenarios*, which are tuples of the form (x, U, V) where (x, U) is an epistemic scenario, $V \in \tau$, and $V \subseteq U$.

- Call V the *doxastic range*.

Weaker notions of belief

The key semantic clauses are:

$$\begin{array}{lll} (x,U,V) \models K\varphi & \text{iff} & U = [\![\varphi]\!]^{U,V} \\ (x,U,V) \models \Box\varphi & \text{iff} & x \in Int([\![\varphi]\!]^{U,V}) \\ (x,U,V) \models B\varphi & \text{iff} & V \subseteq [\![\varphi]\!]^{U,V}, \end{array}$$

where

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- Modalities K and $\Box$ are interpreted (essentially) as before.
- Belief is universal quantification over the doxastic range. Intuitively:
  - V is the agent's "conjecture" about the world, typically stronger than what is guaranteed by her evidence-in-hand U.
  - States in V are considered "more plausible" than the other states in U, so belief = truth in all these more plausible states.

Weaker notions of belief

Note that we do not require that $x \in V$; this corresponds to the intuition that the agent may have false beliefs.

In order to distinguish these semantics from those previous, we refer to them as *epistemic-doxastic (e-d) semantics* for topological subset spaces.

Weaker notions of belief

Theorem 7

$\mathsf{EL}_{K\Box B}$ *is a sound and complete axiomatization of* $\mathcal{L}_{K\Box B}$ *with respect to the class of all topological subset spaces under e-d semantics.*

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Theorem 8

$\mathsf{EL}_{K\Box B} + (\mathsf{wF})$ *is a sound and complete axiomatization of* $\mathcal{L}_{K\Box B}$ *with respect to the class of all topological subset spaces under e-d semantics for dense e-d scenarios.*

Questions?

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