Topological Approaches to Epistemic Logic

Lecture 5: Summary and More on Topological Semantics

Aybüke Özgün

ILLC, University of Amsterdam

Tsinghua Logic Summer School
18.07.2025

Two styles of topological semantics

1. Extensions of interior-based topological semantics
2. (Topological) Subset Space Semantics

Main Motivation

We not only seek an easy way to model knowledge and belief, but also **study the emergence, usage, and transformation of** **evidence** as an **inseparable component of a rational and idealized agent's justified belief and knowledge**.

Interior-based Semantics +

Initial Framework:

Syntax: $\varphi ::= p \mid \varphi \wedge \varphi \mid \neg\varphi \mid K\varphi$

Semantics: Given a topo-model $\mathcal{X} = (X, \tau, V)$, we have

$$[\![K\varphi]\!] = Int([\![\varphi]\!])$$

Interior-based Semantics +

Initial Framework:

Syntax: $\varphi ::= p \mid \varphi \wedge \varphi \mid \neg\varphi \mid K\varphi$

Semantics: Given a topo-model $\mathcal{X} = (X, \tau, V)$, we have

$$[\![K\varphi]\!] = Int([\![\varphi]\!])$$

Q. What about belief?

Belief as co-derivative does not seem to work well!

Interior-based Semantics +

Initial Framework:

Syntax: $\varphi ::= p \mid \varphi \wedge \varphi \mid \neg\varphi \mid K\varphi$

Semantics: Given a topo-model $\mathcal{X} = (X, \tau, V)$, we have

$$[\![K\varphi]\!] = Int([\![\varphi]\!])$$

Q. What about belief?

Belief as co-derivative does not seem to work well!

Next Step:

Syntax: $\varphi ::= p \mid \varphi \wedge \varphi \mid \neg\varphi \mid K\varphi \mid B\varphi$

Semantics: Given a topo-model $\mathcal{X} = (X, \tau, V)$, we have

$$[\![K\varphi]\!] = Int([\![\varphi]\!])$$

$$[\![B\varphi]\!] = Cl(Int([\![\varphi]\!])).$$

Interior-based Semantics +

Initial Framework:

Syntax: $\varphi ::= p \mid \varphi \wedge \varphi \mid \neg\varphi \mid K\varphi$

Semantics: Given a topo-model $\mathcal{X} = (X, \tau, V)$, we have

$$[\![K\varphi]\!] = Int([\![\varphi]\!])$$

Q. What about belief?

Belief as co-derivative does not seem to work well!

Next Step:

Syntax: $\varphi ::= p \mid \varphi \wedge \varphi \mid \neg\varphi \mid K\varphi \mid B\varphi$

Semantics: Given a topo-model $\mathcal{X} = (X, \tau, V)$, we have

$$[\![K\varphi]\!] = Int([\![\varphi]\!])$$

$$[\![B\varphi]\!] = Cl(Int([\![\varphi]\!])).$$

- This works well on extremally disconnected space!
- It sometimes overlaps with belief as co-derivative.
- No syntactic representation of evidence.

Interior-based Semantics +

Next Next Step:

Syntax:

$\varphi := p \mid \neg\varphi \mid \varphi \wedge \varphi \mid E_0\varphi \mid \Box_0\varphi \mid E\varphi \mid \Box\varphi \mid B\varphi \mid K\varphi \mid \forall\varphi$

$E_0\varphi$:= the agent has a *basic (piece of) evidence* supporting φ.

$\Box_0\varphi$:= the agent has a *factive piece of evidence* for φ.

$E\varphi$:= the agent has *(combined) evidence* for φ.

$\Box\varphi$:= the agent has *factive (combined) evidence* for φ.

$B\varphi$:= the agent has a justified belief in φ.

$K\varphi$:= the agent knows φ (in the fallible sense).

$\forall\varphi$:= the agent infallibly knows φ.

Interior-based Semantics +

Next Next Step:
Semantics on Topological Evidence Models: $\mathcal{M} = (X, \mathcal{E}_0, \tau, V)$

Given a topo-e-model $\mathcal{M} = (X, \mathcal{E}_0, \tau, V)$ and $x \in X$, we have

$$\begin{array}{lll}
\mathcal{M}, x \models E_0\varphi & \text{iff} & (\exists e \in \mathcal{E}_0)(e \subseteq [\![\varphi]\!]^{\mathcal{M}}) \\
\mathcal{M}, x \models \Box_0\varphi & \text{iff} & (\exists e \in \mathcal{E}_0)(x \in e \subseteq [\![\varphi]\!]^{\mathcal{M}}) \\
\mathcal{M}, x \models E\varphi & \text{iff} & (\exists e \in \mathcal{E})(e \subseteq [\![\varphi]\!]^{\mathcal{M}}) \\
\mathcal{M}, x \models \Box\varphi & \text{iff} & (\exists e \in \mathcal{E})(x \in e \subseteq [\![\varphi]\!]^{\mathcal{M}}) \\
\mathcal{M}, x \models B\varphi & \text{iff} & Cl(Int([\![\varphi]\!]^{\mathcal{M}})) = X \\
\mathcal{M}, x \models K\varphi & \text{iff} & x \in Int([\![\varphi]\!]^{\mathcal{M}}) \text{ and } Cl(Int([\![\varphi]\!]^{\mathcal{M}})) = X \\
\mathcal{M}, x \models \forall\varphi & \text{iff} & [\![\varphi]\!] = X
\end{array}$$

★ *$\mathcal{E}_0$ represents the set of evidence pieces the agent has already acquired about the actual situation.*

Interior-based Semantics +

We have focused on

- Motivation behind using such semantics.
- Soundness and completeness results with respect to (sometimes restricted) classes of topological spaces.
- Discussion in Epistemology: Gettier counterexamples

Interior-based Semantics $+$: what we have not done

- Conditional Beliefs and Dynamics of Evidence Management
  public announcements, evidence addition, evidence upgrade, (feasible) evidence combination...

van Benthem, J. and Pacuit, E. *Dynamic Logics of Evidence-Based Beliefs*. Studia Logica (2011) 99: 61.

Özgün, A. (2017) *Evidence in Epistemic Logic: A topological perspective*. PhD thesis. Université de Lorraine & University of Amsterdam - Chapter 5.

Baltag, A., Bezhanishvili, N., Özgün, A., and Smets. (2012). *Justified belief, knowledge, and the topology of evidence*. Synthere, 200(6):1-51.

Interior-based Semantics +: what we have not done

- Multi-agent extensions
  Which topological constructions are appropriate? Product spaces, Sums, multiple topologies on one domain....
  How to define group notions of knowledge and belief? How the agent should/can pool their evidence together?

dos Santos Gomes, D. (2025) *Virtual Group Knowledge on Topological Evidence Models*. Master's thesis, ILLC, University of Amsterdam.

Baltag, A. and Liberman, A. O. (2017) *Evidence Logics with Relational Evidence*. Proceedings of LORI 2017: 17-32.

Fernańdez González, S. (2018) *Generic Models for Topological Evidence Logics.* Master's thesis, ILLC, University of Amsterdam.

Liberman, A. O. (2016). *Dynamic Evidence Logics with Relational Evidence.* Master's thesis, ILLC, University of Amsterdam.

Ramirez, A. I. R. (2015) *Topological models for group knowledge and belief.* Master's thesis, ILLC, University of Amsterdam.

Interior-based Semantics $+$: what we have not done

From Epistemic Logic to Abstract Modal Logic

For example...

Theorem (McKinsey and Tarski, 1944)

- $\mathsf{S4}_K$ *is complete wrt all topological spaces.*

Interior-based Semantics $+$: what we have not done

For example...

Theorem (McKinsey and Tarski, 1944)

- $\mathsf{S4}_K$ *is complete wrt all topological spaces.*
- $\mathsf{S4}_K$ *is complete wrt the real line* $\mathbb{R}$.
- $\mathsf{S4}_K$ *is complete wrt any Euclidean space* $\mathbb{R}^n$.

For example...

Theorem (McKinsey and Tarski, 1944)

- $\mathsf{S4}_K$ *is complete wrt all topological spaces.*
- $\mathsf{S4}_K$ *is complete wrt the real line* $\mathbb{R}$.
- $\mathsf{S4}_K$ *is complete wrt any Euclidean space* $\mathbb{R}^n$.

- Completeness results with respect to specific topological spaces

van Benthem J., Bezhanishvili G. (2007) *Modal Logics of Space.* In: Aiello M., Pratt-Hartmann I., Van Benthem J. (eds) Handbook of Spatial Logics. Springer, Dordrecht.

Baltag, A., Bezhanishvili, N. and Fernańdez González, S. (2019) *The McKinsey-Tarski Theorem for Topological Evidence Logics*. Proceedings of WoLLIC 2019: 177-194.

Fernańdez González, S. (2018) *Generic Models for Topological Evidence Logics.* Master's thesis, ILLC, University of Amsterdam.

Interior-based Semantics +: what we have not done

- Even more work....

Baltag, A. and van Benthem, J. (2025) *Knowability as Continuity: a topological account of informational dependence*. Logics 3(3), 6.

Baltag, A., Fernandez-Duque, D. and Bezhanishvili, N. (2022) *The Topology of Surprise*. Awarded Ray Reiter Best Paper Award for research in AI at the KR (Knowledge Representation) 2022.

Baltag, A., Fernandez-Duque, D. and Bezhanishvili, N. (2021) *The Topological Mu-Calculus: completeness and decidability*, LICS 2021.

Dekker, P. M. (2023). *KD45 with Propositional Quantifiers.* Logic and Logical Philosophy, 33(1), 27–54. https://doi.org/10.12775/LLP.2023.018

Gougeon, Q. (2024) *Some completeness results in derivational modal logic*. Journal of Logic and Computation, Volume 34, Issue 7, October 2024, Pages 1211–1248, https://doi.org/10.1093/logcom/exad047

Steinsvold, C. (2020). *Some Formal Semantics for Epistemic Modesty*. Logic and Logical Philosophy, 29(3), 381–413. https://doi.org/10.12775/LLP.2020.002

(Topological) Subset Space Semantics

Initial Framework:

Syntax: $\varphi := p \mid \varphi \wedge \varphi \mid \neg\varphi \mid K\varphi \mid \blacksquare\varphi$

Semantics: Given a subset space model $\mathcal{X} = (X, \mathcal{O}, \nu)$ and an epistemic scenario (x, U) of $\mathcal{X}$,

(Topological) Subset Space Semantics

Initial Framework:

Syntax: $\varphi := p \mid \varphi \wedge \varphi \mid \neg\varphi \mid K\varphi \mid \blacksquare\varphi$

Semantics: Given a subset space model $\mathcal{X} = (X, \mathcal{O}, \nu)$ and an epistemic scenario (x, U) of $\mathcal{X}$,

$$\begin{array}{lll} \mathcal{X}, (x,U) \models K\varphi & \text{iff} & (\forall y \in U)(\mathcal{X}, (y,U) \models \varphi) \\ \mathcal{X}, (x,U) \models \blacksquare\varphi & \text{iff} & \forall V \in \mathcal{O}(x \in V \subseteq U \Rightarrow \mathcal{X}, (x,V) \models \varphi) \end{array}$$

★ *$\mathcal{O}$ represents the set of potential evidence the agent can in principle discover, even if she does not happen to personally have it in hand at the moment.*

★ *Given an epistemic scenario (x, U), x represents the actual world and U the agent's current evidence.*

(Topological) Subset Space Semantics

Next Step: Knowledge, Knowability, and Belief

Syntax: $\varphi := p \mid \neg\varphi \mid \varphi \wedge \varphi \mid K\varphi \mid \Box\varphi \mid B\varphi$

Semantics: Given a topological model $\mathcal{X} = (X, \tau, \nu)$ and an epistemic scenario (x, U) of $\mathcal{X}$,

$$
\begin{array}{lll}
(x,U) \models K\varphi & \text{iff} & (\forall y \in U)((y,U) \models \varphi) \\
(x,U) \models \Box\varphi & \text{iff} & x \in Int([\![\varphi]\!]^U) \\
(x,U) \models B\varphi & \text{iff} & U \subseteq Cl(Int([\![\varphi]\!]^U))
\end{array}
$$

(Topological) Subset Space Semantics

Next Step: Knowledge, Knowability, and Belief

Syntax: $\varphi := p \mid \neg\varphi \mid \varphi \wedge \varphi \mid K\varphi \mid \Box\varphi \mid B\varphi$

Semantics: Given a topological model $\mathcal{X} = (X, \tau, \nu)$ and an epistemic scenario (x, U) of $\mathcal{X}$,

$$
\begin{array}{lll}
(x,U) \models K\varphi & \text{iff} & (\forall y \in U)((y,U) \models \varphi) \\
(x,U) \models \Box\varphi & \text{iff} & x \in Int([\![\varphi]\!]^U) \\
(x,U) \models B\varphi & \text{iff} & U \subseteq Cl(Int([\![\varphi]\!]^U))
\end{array}
$$

- Although motivated independently, we ended up with belief as *dense interior* in subset space-style semantic.
- We also had a look at weaker notions of belief on topological subset space semantics.

(Topological) Subset Space Semantics: we have not done

- Knowability and public announcements
  $\varphi := p \mid \neg\varphi \mid \varphi \wedge \varphi \mid K\varphi \mid \Box\varphi \mid [\varphi]\varphi$

  Bjorndahl, A. (2022) *Topological subset space models for public announcements.* In van Ditmarsch, H. and Sandu, G., editors, Jaakko Hintikka on Knowledge and Game-Theoretical Semantics, pages 165–186, Cham. Springer International Publishing.

- Topo-Logic with knowability and public announcements
  $\varphi := p \mid \neg\varphi \mid \varphi \wedge \varphi \mid K\varphi \mid \Box\varphi \mid [\varphi]\varphi \mid \blacksquare\varphi$

  Baltag A., Özgün A., Vargas Sandoval A.L. (2017) *Topo-Logic as a Dynamic-Epistemic Logic*. Proceedings of LORI 2017, pp. 330-346.

(Topological) Subset Space Semantics: we have not done

- Dynamic Logics for Formal Learning Theory
  Formalizing important notions of Formal Learning Theory, such as inductive verifiability, falsifiability, knowledge, learning, by using modal logic and subset space semantics.

Baltag A., Gierasimczuk N., Özgün A., Vargas Sandoval A.L. and Smets S. (2019) *A Dynamic Logic for Learning Theory.* Journal of Logical and Algebraic Methods in Programming, volume 109.

Baltag A., Özgün A., Vargas Sandoval A.L. (2019) *The Logic of AGM Learning from Partial Observations*. To appear in Dynamic Logic. New Trends and Applications. DALI 2019.

(Topological) Subset Space Semantics: we have not done

- Dynamic Logics for Formal Learning Theory
  Formalizing important notions of Formal Learning Theory, such as inductive verifiability, falsifiability, knowledge, learning, by using modal logic and subset space semantics.

  For further references on the connection between Topology, Formal Learning Theory, and Logic please see:

Baltag, A., Gierasimczuk, N., and Smets, S. (2011) *Belief revision as a truth-tracking process.* Proceedings of TARK 2011, pp. 187-190.

Baltag, A., Gierasimczuk, N., and Smets, S. (2015) *On the solvability of inductive problems: A study in epistemic topology.* Proceedings of TARK 2015, pp. 81-98.

Kelly, K.T. *The Logic of Reliable Inquiry,* Oxford University Press, 1996.

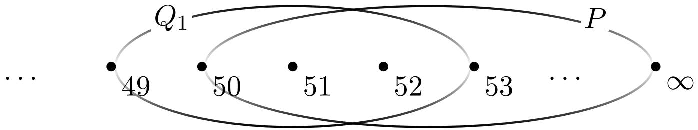
(Topological) Subset Space Semantics: we have not done

- Uncertainty about Evidence
  Modelling agents who are uncertain about how to interpret their evidence.

Example: speed of a car

A policeman uses a radar gun with accuracy ± 2 mph to determine whether a car is speeding in a 50 mph speed-limit zone. Suppose the radar gun shows 51 mph:

$P = (50, \infty) :=$ the car is speeding
$Q_1 = (49, 53) :=$ the reading of the 1st-radar gun is 51 mph

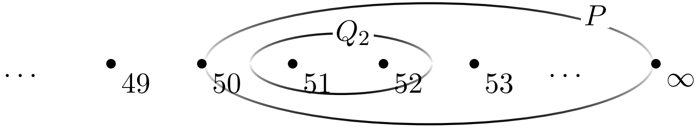


With the measurement $(49, 53)$ in hand, the policeman cannot be said to *know* that the car is speeding: $(49, 53) \not\subseteq (50, \infty)$.

Example: speed of a car

If the policeman takes another measurement using a more accurate radar gun with an accuracy of ± 1 mph which shows 51.5 mph.

$P = (50, \infty) :=$ the car is speeding
$Q_2 = (50.5, 52.5) :=$ the reading of the 2nd-radar gun is 51.5 mph



With the measurement $(50.5, 52.5)$ in hand, the policeman *knows* that the car is speeding: $(50.5, 52.5) \subseteq (50, \infty)$.

Example: speed of a car

After having taken such measurements, the policeman is still *uncertain* about the actual speed of the car since his evidence is imprecise.

But, the evidence itself is certain in the sense that he knows what it *actually* entails (he knows the margin of error).

BUT, we may take a measurement with a certain margin of error without being sure of what exactly that margin is!

(Topological) Subset Space Semantics: we have not done

- Uncertainty about Evidence
  Modelling agents who are uncertain about how to interpret their evidence.

  Bjorndahl, A. and Özgün, A. (2019) *Uncertainty about Evidence*. Proceedings of TARK 2019, pp. 68-81.

(Topological) Subset Space Semantics: we have not done

- Multi-agent extensions....

(Topological) Subset Space Semantics: we have not done

- Technical results and further extensions of subset space logics
  More completeness, decidability, complexity.

  In addition to the sources we have on slide 29:

  Parikh R., Moss L., Steinsvold C. (2007) *Topology and Epistemic Logic.* In: Aiello M., Pratt-Hartmann I., van Benthem J. (eds) Handbook of Spatial Logics. Springer, Dordrecht.

In general, Handbook of Spatial Logics is a great source.

It is impossible list all the relevant papers here. We hope that the list of references we have provided will guide you through the literature.

Announcement: Workshop on Monday, July 21, 2025

The 17th Tsinghua Logic Colloquium:
Advances in Philosophical Logic

Time: 13:30-17:30, July 21th, 2025
Venue: Room 329, School of Humanities, Tsinghua University.

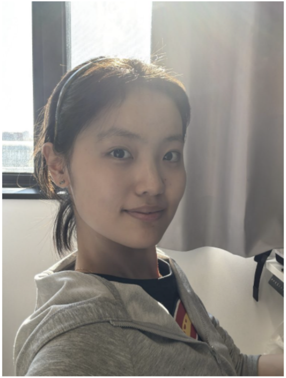
Website: `http://tsinghualogic.net/JRC/the-17th-tsinghua-logic-colloquium/`

The material presented at this course is based on several joint papers with Alexandru Baltag, Nick Bezhanishvili, Adam Bjorndahl, Hans van Ditmarsch, Nina Gierasimczuk, Sophia Knight, Sonja Smets, and Ana Lucia Vargas Sandoval.

I thank them for their contribution to this course (by, at the very least, having spent a lot of time discussing topological semantics with me and providing slides).

Special thanks to Alexandru Baltag for sharing the slides of his course *Topology, Logic, and Learning*.

THANKS!



Enjoy the party!

(and don't forget to submit your final exam on July 22nd :)!)