Topological Approaches to Epistemic Logic
Final Exam

Due Tuesday, July 22, 2025, 23h59 (Beijing time)!

The homework is **individual** and **obligatory** for passing the course.

1. (20 pts) Let $\mathfrak{M} = (X, \mathcal{E}_0, \tau, V)$ be a topo-e-model and τ_{dense} be the corresponding dense topology defined as
$$\tau_{dense} = \{U \in \tau \mid Cl(U) = X\} \cup \{\emptyset\}$$
(recall from HW2-Q2).

   Check our definitions of justified belief B and (fallible) knowledge K given in Lecture 3: for any $P \subseteq X$, we have
$$x \in KP \text{ iff } x \in Int(P) \text{ and } Cl(Int(P)) = X,$$
$$x \in BP \text{ iff } Cl(Int(P)) = X.$$
   Show that

   (a) (10 pts) $KP = Int_{dense}(P)$, and

   (b) (10 pts) $BP = Cl_{dense}(Int_{dense}(P))$, where Int_{dense} and Cl_{dense} are the interior and closure operators of τ_{dense}, respectively.

   Note: If useful, you may use some of the other homework questions in your answer.

2. (60 pts) For this exercise, we work with the unimodal language $\mathcal{L}_B$ and define an alternative topological semantics for B.

   Given a topo-model $\mathcal{X} = (X, \tau, V)$, the semantics of $B\varphi$ is given as
$$[\![B\varphi]\!]^{\mathcal{X}} = Int(Cl(Int([\![\varphi]\!]^{\mathcal{X}}))).$$
   In this exercise, you are asked to show (a)-(c) given below for the system weak-$\mathsf{KD45}_B$ given in Table 1. Read Definition 1 and Lemma 1 carefully and use them in your answer as needed (you *do not* need to prove Lemma 1):

   Definition 1. *Given a Kripke frame* (X, R)*:*

	$\mathsf{wKD45}_B$ axioms & rules	
(K_B)	$B(\varphi \wedge \psi) \leftrightarrow (B\varphi \wedge B\psi)$	Closure under conjunction
(D_B)	$B\varphi \to \neg B \neg \varphi$	Consistency
(4_B)	$B\varphi \to BB\varphi$	Positive Introspection
$(\mathrm{w}5_B)$	$B\widehat{B}B\varphi \to B\varphi$	weak Negative Introspection
(Nec)	from ϕ, infer $B\varphi$	

Table 1: $\mathsf{wKD45}_B$

- *a nonempty $C \subseteq X$ is called a* **cluster** *if for all $x, y \in C$, xRy.*
- *$C \subseteq X$ is called a* **maximal cluster** *if it is a cluster and for all $x, y \in X$ if $x \in C$ and xRy then $y \in C$.*
- *(X, R) is called a* **rooted weak pin** *if there is a unique $x \in X$ and a family of disjoint maximal clusters $\{C_i\}_{i \in I}$ such that for all $y \in \bigcup_{i \in I} C_i$, $(x, y) \in R$ and $(y, x) \notin R$ (y is not related to x via R), and $X = \bigcup_{i \in I} C_i \cup \{x\}$.*

In other words, a rooted weak pin is a Kripke frame of the form given in Figure 1, where the round-squares represent maximal clusters C_i.

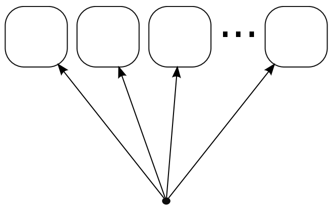

Figure 1:
Rooted pin

Lemma 1. *$wKD45$ is sound and complete w.r.t. the class of finite rooted weak pins.*

Prove the following:

(a) (20 pts) Axioms K_B and D_B are valid in all topo-models under the new semantics.

(b) (10 pts) Also show that the negative introspection axiom for belief, namely $\neg B\varphi \to B\neg B\varphi$, is not valid in all topo-models under the new semantics.

Note: Prove (a) and (b) directly by using the semantics given above.

(c) (30 pts) [More challenging] Prove that $\mathsf{wKD45}$ is complete wrt the class of all topo-models.

Hint: For completeness, see the completeness proof for $\mathsf{KD45}$ under the closure of the interior semantics in [1, Appendix A5]. Follow similar steps; modify the proof for the new system and the new semantics. In particular, given a finite rooted weak pin, construct a topological space and prove Lemma 7 and Theorem 7 in [1, Appendix A5] for the new semantics. *Show that the topology you constructed is indeed a topological space.* Use Lemma 1 given above in your completeness proof.

3. (20 pts) For this exercise we work with the subset space semantics introduced in Lecture 4. Recall the axiom (AP) on slide 23 of Lecture 4:

   (AP) $(p \to \blacksquare p) \wedge (\neg p \to \blacksquare \neg p)$ for all $p \in Prop$.

   (a) (10 pts) Show that for every φ in the language of classical propositional logic (elements of $\mathcal{L}_{K\blacksquare}$ that do not have occurrences of K or $\blacksquare$),

   $$(\varphi \to \blacksquare\varphi) \wedge (\neg\varphi \to \blacksquare\neg\varphi)$$

   is valid in all subset space models.

   (b) (10 pts) Show that the validity in (a) does not hold for all $\psi \in \mathcal{L}_{K\blacksquare}$. That is, find a $\psi \in \mathcal{L}_{K\blacksquare}$ such that $(\psi \to \blacksquare\psi) \wedge (\neg\psi \to \blacksquare\neg\psi)$ is not valid and justify your answer.

   Note: If useful, you may use some of the other homework questions in your answer.

References

[1] Alexandru Baltag, Nick Bezhanishvili, Aybüke Özgün, and Sonja Smets. A topological approach to full belief. *Journal of Philosophical Logic*, 48(2):205–244, Apr 2019.