Topological Approaches to Epistemic Logic
Homework 2

Due Wednesday, July 17, 2025, 13h30 (Beijing time)!

The homework is **individual** and **obligatory** for passing the course. BONUS question is not mandatory but it will give you extra points.

1. (30 pts) Given a topological space (X, τ) and any two subsets $U_1, U_2 \subseteq X$
   (a) (15 pts) if U_1 is open and dense and U_2 is dense, then $U_1 \cap U_2$ is dense.
   (b) (15 pts) if U_1 and U_2 are nowhere-dense, then $U_1 \cup U_2$ are nowhere-dense.
2. (50 pts) Let (X, τ) be a topological space and define τ_{dense} on X as follows:
   $$\tau_{dense} = \{U \in \tau \mid Cl(U) = X\} \cup \{\emptyset\},$$
   where Cl is the closure operator of τ. Prove that
   (a) (10 pts) for all nonempty $U, V \in \tau_{dense}$, we have $U \cap V \neq \emptyset$,
   (b) (20 pts) τ_{dense} is a topology,
   (c) (20 pts) τ_{dense} is extremally disconnected.
3. (20 pts) As stated on slide 50 of Lecture 2, the logic $\mathsf{KD45}_B$ is sound and complete with respect to the class of extremally disconnected spaces (under the closure of interior semantics). Which axioms of $KD45_B$ are invalidated by some topo-models (that are not extremally disconnected) when we interpret $B\varphi$ as $[\![B\varphi]\!] = Cl(Int([\![\varphi]\!]^{\mathcal{X}}))$? Justify your answers by giving counterexamples that invalidate these axioms and explaining why the axioms are false in these models.
4. (BONUS - 20 pts) Read the following definitions carefully:

   Definition 1 (Subspace)**.** *Given a topological space (X, τ) and a nonempty subset $P \subseteq X$, the topological space (P, τ_P) is called a* subspace *of (X, τ) (*induced by *P), where*
   $$\tau_P = \{U \cap P \mid U \in \tau\}.$$

   Definition 2. *A topological space (X, τ) is called* hereditarily extremally disconnected *if every subspace of (X, τ) is extremally disconnected.*

   Let (X, R) be a total preordered[1] set. Show that the Alexandroff topology τ_R, defined as
   $$\tau_R = \{U \subseteq X \mid U \text{ is an up-set of } (X, R)\},$$
   is hereditarily extremally disconnected.

[1] See the course handout for the definition of total preorders