Topological Approaches to Epistemic Logic
Homework 3

Due Wednesday, July 18, 2025, 13h30 (Beijing time)!

The homework is **individual** and **obligatory** for passing the course. BONUS question is not mandatory but it will give you extra points.

1. (60 pts)For this exercise we work with the topological subset space semantics introduced in Lecture 4. Read the following definition carefully:

   Definition 1 (Persistent in subset spaces)**.** *A formula* $\varphi \in \mathcal{L}_{K\blacksquare}$ *is called* persistent *(with respect to the subset space semantics) if for all subset space models* $\mathcal{X} = (X, \mathcal{O}, V)$ *and for all epistemic scenarios* (x, U) *and* (x, O) *of* $\mathcal{X}$*, we have* $(x, U) \models \varphi$ *iff* $(x, O) \models \varphi$*.*

   Show that

   (a) (20 pts) Every φ in the language of classical propositional logic (elements of $\mathcal{L}_{K\blacksquare}$ that do not have occurrences of K or $\blacksquare$) is persistent with respect to the subset space semantics.

   (b) (20 pts) Show that not every formula in $\mathcal{L}_{K\blacksquare}$ is persistent. Justify your answer by finding a counterexample and showing that it is not persistent.

   (c) (20 pts) Are there any persistent formulas **with respect to topological subset spaces models** in $\mathcal{L}_{K\blacksquare}$ that *have occurrences of* K *or* $\blacksquare$? That is, is there a formula $\psi \in \mathcal{L}_{K\blacksquare}$ that has some occurrences of K or $\blacksquare$ such that for all **topological models** $\mathcal{X} = (X, \tau, V)$ and for all epistemic scenarios (x, U) and (x, O) of $\mathcal{X}$, we have $(x, U) \models \psi$ iff $(x, O) \models \psi$? Justify your answer either by given an example of such a persistent formula and proving your claim; or by proving that only the formulas in the language of classical propositional logic are persistent.

2. (40 pts) For this exercise we work with the topological subset space semantics introduced in Lecture 4. Given a topo-model $\mathcal{X} = (X, \tau, V)$ and $p \in Prop$, prove the following:

   (a) (20 pts) $V(p)$ is an open set of (X, τ) iff $p \to \blacklozenge Kp$ is valid in $\mathcal{X}$.

   (b) (20 pts) $V(p)$ is a closed set of (X, τ) iff $\blacksquare \hat{K} p \to p$ is valid in $\mathcal{X}$.

   Note: Use the definition of validity on slide 22 of Lecture 4.

3. (BONUS - 20 pts) For this exercise, we work with topo-e-models and the corresponding semantics introduced in the slides of Lecture 3.

By adding a piece of evidence $P \subseteq X$ to a topo-e-model $\mathfrak{M} = (X, \mathcal{E}_0, \tau, V)$, we can create another topo-e-model $\mathfrak{M}^{+P}$. Let us define $\mathfrak{M}^{+P}$ as $\mathfrak{M}^{+P} = (X, \mathcal{E}_0^{+P}, \tau^{+P}, V)$, where $\mathcal{E}_0^{+P} = \mathcal{E}_0 \cup \{P\}$ and τ^{+P} is the topology generated by $\mathcal{E}_0^{+P}$.

We now introduce a new modality, $[+\varphi]\psi$, into our language $\mathcal{L}$ (on slide 51 of Lecture 3) and interpret it in a given topo-e-model $\mathfrak{M} = (X, \mathcal{E}_0, \tau, V)$ at $x \in X$ as follows:

$$\mathfrak{M}, x \in [\![[+\varphi]\psi]\!]^{\mathfrak{M}} \text{ iff } [\![\varphi]\!]^{\mathfrak{M}} \neq \emptyset \text{ implies } x \in [\![\psi]\!]^{\mathfrak{M}^{+\varphi}}.$$

Show that the following formula is valid in all topo-e-models:

- $[+\varphi]\Box\psi \leftrightarrow ([\exists]\varphi \to (\Box[+\varphi]\psi \vee (\varphi \wedge \Box(\varphi \to [+\varphi]\psi))))$