Topological Approaches to Epistemic Logic
-
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Contents

NOTE: These lecture notes are based on the material in Aybüke Özgün's PhD dissertation (Özgün, 2017) and the course *Topological Approaches in Epistemic Logic*, co-taught by Sophia Knight, at ESSLLI 2016. The reader who is interested in proof details and further discussion on the material presented in this draft can find the full dissertation here: `http://www.illc.uva.nl/Research/Publications/Dissertations/DS-2017-07.text.pdf`.

This handout presents background material for the course. The lectures will go beyond the material presented here. Further relevant literature will be provided on course slides.

1 Background

In this section, we provide the technical preliminaries essential for the class. This course focuses primarily on topological semantics for (dynamic) epistemic and doxastic logics, however, we occasionally resort to their connection with the relational semantics and the well-developed completeness results therein, in order to obtain similar conclusions for the topological counterpart. The basics of the course rely on three different formal settings: the standard relational semantics for the basic modal logic, the interior-based topological semantics à la McKinsey & Tarski (1944), and the subset space semantics introduced by Moss & Parikh (1992). While the relational setting serves only as a technical tool utilized in obtaining meta-logical results, the latter two topological settings have inspired several logics that formalize, besides various notions of knowledge and belief, important notions such as *justification, evidence, argument,* and *knowability.* We leave the conceptual motivation for using topological semantics for the classroom, and present here only the background formal tools.

1.1 Relational Semantics for Modal Logic

In this section, we briefly present the standard relational semantics for the basic modal language and define some well-known epistemic and doxastic logics. This is in no way an exhaustive presentation of relational semantics for modal epistemic and doxastic logics: here we aim to fix notation and summarize the results we use during the lecture. For a more comprehensive treatment of modal logic, please consult, e.g., Blackburn et al. (2001); Chagrov & Zakharyaschev (1997). The presentation in this section is based on the basic unimodal modal language since we make use of the technical aspects of the relational setting to prove results almost exclusively regarding unimodal epistemic/doxastic systems.

Definition 1 (Syntax of $\mathcal{L}_\Box$)**.** *The language of* basic modal logic $\mathcal{L}_\Box$ *is defined recursively as*

$$\varphi ::= p \mid \neg\varphi \mid (\varphi \wedge \varphi) \mid \Box\varphi,$$

where $p \in \mathsf{Prop}$*, a countable set of* propositional variables*.*

Abbreviations for the Boolean connectives $\vee, \rightarrow$ and $\leftrightarrow$ are standard, and $\bot$ is defined as $p \wedge \neg p$. We employ $\Diamond\varphi$ as an abbreviation for $\neg\Box\neg\varphi$. We follow the usual rules for elimination of parenthesis in the language.

Since we, in general, work with the above defined modal language in an epistemic/doxastic setting, the particular languages we consider in this work typically include, instead of $\Box$, modalities such as K and B for knowledge and belief, respectively. Accordingly, $\mathcal{L}_K$ denotes the *basic epistemic language* and $\mathcal{L}_B$ the *basic doxastic language* defined as in Definition 1.

We are particularly interested in the modal systems that are commonly used in the formal epistemology literature to represent notions of knowledge and belief. Some of the interesting and widely used axioms and an inference rule formalizing properties of these notions are listed in Table 1.

We again use a similar notational convention as we did in case of the languages. For example, the axiom of Consistency for *belief* is denoted by (D_B) $B\varphi \rightarrow \neg B\neg\varphi$, Positive Introspection for *knowledge* is written as (4_K) $K\varphi \rightarrow KK\varphi$, etc.

Let CPL denote all instances of classical propositional tautologies (see, e.g., Chagrov & Zakharyaschev, 1997, Section 1.3 for an axiomatization of classical propositional logic). Throughout

($K_\Box$)	$\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$	Normality
($D_\Box$)	$\Box\varphi \to \neg\Box\neg\varphi$	Consistency
($T_\Box$)	$\Box\varphi \to \varphi$	Factivity
($4_\Box$)	$\Box\varphi \to \Box\Box\varphi$	Positive Introspection
($.2_\Box$)	$\neg\Box\neg\Box\varphi \to \Box\neg\Box\neg\varphi$	Directedness
($.3_\Box$)	$\Box(\Box\varphi \to \psi) \vee \Box(\Box\psi \to \varphi)$	Connectedness
($5_\Box$)	$\neg\Box\varphi \to \Box\neg\Box\varphi$	Negative Introspection
($Nec_\Box$)	from φ, infer $\Box\varphi$	Necessitation
(MP)	from $\varphi \to \psi$ and φ, infer $\vdash \psi$	Modus Ponens

Table 1: Some unimodal axiom schemes and a rule of inference for $\Box$

these notes, we use Hilbert-style axiom systems in order to provide the syntactic definitions of the modal logics we work with. Recall that, the weakest/smallest *normal* modal logic, denoted by $\mathsf{K}_\Box$, is defined as the least subset of $\mathcal{L}_\Box$ containing all instances of propositional tautologies (CPL) and ($K_\Box$), and closed under the inference rules (MP) and ($Nec_\Box$). Then, following standard naming conventions, we define the following normal modal logics that are used to represent knowledge and belief of agents with different reasoning power, where $\mathsf{L}+(\varphi)$ denotes the smallest modal logic containing L and φ. In other words, $\mathsf{L}+(\varphi)$ is the smallest set of formulas (in the corresponding language) that contains L and φ, and is closed under the inference rules of L. For example:

$$
\begin{array}{rcl}
\mathsf{KT}_\Box & = & \mathsf{K}_\Box + (T_\Box) \\
\mathsf{S4}_\Box & = & \mathsf{KT}_\Box + (4_\Box) \\
\mathsf{S4.2}_\Box & = & \mathsf{S4}_\Box + (.2_\Box) \\
\mathsf{S4.3}_\Box & = & \mathsf{S4}_\Box + (.3_\Box) \\
\mathsf{S5}_\Box & = & \mathsf{S4}_\Box + (5_\Box) \\
\mathsf{KD45}_\Box & = & \mathsf{K}_\Box + (D_\Box) + (4_\Box) + (5_\Box)
\end{array}
$$

Table 2: Some normal (epistemic/doxastic) modal logics

While the systems $\mathsf{S4}_K, \mathsf{S4.2}_K, \mathsf{S4.3}_K$ and $\mathsf{S5}_K$ are considered to be logics for knowledge of different strength, much work on the formal representation of belief takes the logical principles of $\mathsf{KD45}_B$ for granted (see, e.g., Baltag et al. (2008); van Ditmarsch et al. (2007); Baltag & Smets (2008)). Hintikka (1962) considered $\mathsf{S4}_K$ to be the logic of knowledge, $\mathsf{S4.2}_K$ is defended by Lenzen (1978) and Stalnaker (2006). van der Hoek (1993); Baltag & Smets (2008) studied $\mathsf{S4.3}_K$ as epistemic logics for agents of stronger reasoning power. While the system $\mathsf{S5}_K$ is used in applications of logic in computer science Fagin et al. (1995); Meyer & van der Hoek (1995); van Ditmarsch et al. (2007), it is, as a logic of knowledge, often deemed to be too strong and rejected by philosophers (see, e.g., Hintikka, 1962; Voorbraak, 1993, for arguments against $\mathsf{S5}_K$). Throughout the course, all these systems will occasionally recur within different topological frameworks. In the following, we first present their standard relational semantics.

Before moving on to the standard relational semantics for the basic modal logic, we briefly recall the following standard terminology for Hilbert-style axiom systems, and set some notation. Given a logic L defined by a (finitary) Hilbert-style axiom system, an L-*derivation/proof* is a finite sequence of formulas such that each element of the sequence is either an axiom of L, or obtained from the previous formulas in the sequence by one of the inference rules. A formula φ is called

L-*provable*, or, equivalently, a *theorem* of L, if it is the last formula of some L-proof. In this case, we write $\vdash_{\mathsf{L}} \varphi$ (or, equivalently, $\varphi \in \mathsf{L}$). For any set of formulas Γ and any formula φ, we write $\Gamma \vdash_{\mathsf{L}} \varphi$ if there exist finitely many formulas $\varphi_1, \ldots, \varphi_n \in \Gamma$ such that $\vdash_{\mathsf{L}} (\varphi_1 \wedge \cdots \wedge \varphi_n) \rightarrow \varphi$. We say that Γ is L-*consistent* if $\Gamma \nvdash_{\mathsf{L}} \bot$, and L-*inconsistent* otherwise. A formula φ is *consistent with* Γ if $\Gamma \cup \{\varphi\}$ is L-consistent (or, equivalently, if $\Gamma \nvdash_{\mathsf{L}} \neg\varphi$). Finally, a set of formulas Γ is *maximally consistent* if it is L-consistent and any set of formulas properly containing Γ is L-inconsistent, i.e. Γ cannot be extended to another L-consistent set. We drop mention of the logic L when it is clear from the context.

Definition 2 (Relational Frame/Model)**.** *A* relational frame $\mathcal{F} = (X, R)$ *is a pair where X is a nonempty set and $R \subseteq X \times X$. A* relational model $\mathcal{M} = (X, R, V)$ *is a tuple where (X, R) is a relational frame and $V : \mathsf{Prop} \rightarrow \mathcal{P}(X)$ is a valuation map.*

Relational frames/models are also called *Kripke frames/models*. We use these names interchangeably. We say $\mathcal{M} = (X, R, V)$ is a relational model *based on the frame* $\mathcal{F} = (X, R)$. While elements of X are called *states* or *possible worlds*, one of which represents the actual state of affairs, called the *actual* or *real* state, R is known as the *accessibility* or *indistinguishability relation*. We let $R(x) = \{y \in X \mid xRy\}$. The set $R(x)$ represents the set of states that the agent considers possible at x. This way a relational structure models the agent's *uncertainty* about the actual situation. The semantics of $\mathcal{L}_\Box$ on relational models is given below.

Definition 3 (Relational Semantics for $\mathcal{L}_\Box$)**.** *Given a relational model $\mathcal{M} = (X, R, V)$ and a state $x \in X$,* truth *of a formula in the language $\mathcal{L}_\Box$ is defined recursively as follows:*

$$
\begin{array}{lll}
\mathcal{M}, x \models p & \textit{iff} & x \in V(p), \ \textit{where } p \in \mathsf{Prop} \\
\mathcal{M}, x \models \neg\varphi & \textit{iff} & \textit{not } \mathcal{M}, x \models \varphi \\
\mathcal{M}, x \models \varphi \wedge \psi & \textit{iff} & \mathcal{M}, x \models \varphi \textit{ and } \mathcal{M}, x \models \psi \\
\mathcal{M}, x \models \Box\varphi & \textit{iff} & \textit{for all } y \in X, \ \textit{if } xRy \textit{ then } \mathcal{M}, y \models \varphi.
\end{array}
$$

It follows from the above definition that

$$\mathcal{M}, x \models \Diamond\varphi \quad \text{iff} \quad \text{there is } y \in X \text{ such that } xRy \text{ and } \mathcal{M}, y \models \varphi.$$

We adopt the standard notational conventions and abbreviations (see e.g., Blackburn et al., 2001, Chapter 1.3). If $\mathcal{M}$ does not make φ true at x, we write $\mathcal{M}, x \not\models \varphi$. In this case, we say that φ is *false* at x in $\mathcal{M}$. When the corresponding model is clear from the context, we write $x \models \varphi$ for $\mathcal{M}, x \models \varphi$.

We call a formula φ *valid in a relational model* $\mathcal{M} = (X, R, V)$, denoted by $\mathcal{M} \models \varphi$, if $\mathcal{M}, x \models \varphi$ for all $x \in X$, and it is *valid in a relational frame* $\mathcal{F} = (X, R)$, denoted by $\mathcal{F} \models \varphi$, if $\mathcal{M} \models \varphi$ for every relational model based on $\mathcal{F}$. Moreover, we say φ is *valid in a class $\mathcal{K}$ of relational frames*, denoted by $\mathcal{K} \models \varphi$, if $\mathcal{F} \models \varphi$ for every member of this class, and it is *valid*, denoted by $\models \varphi$, if it is valid in the class of all frames. These definitions can easily be extended to sets of formulas in the following way: a set $\Gamma \subseteq \mathcal{L}_\Box$ is *valid in a relational frame* $\mathcal{F}$ iff $\mathcal{F} \models \varphi$ for all $\varphi \in \Gamma$. These definitions also apply to class of models in the same way. More general, we can define a notion of *logical consequence over a class of frames/models* as follows (this notion is called *local semantic consequence* in (Blackburn et al., 2001, Chapter 1.5)). Given $\Gamma \subseteq \mathcal{L}_\Box$, $\varphi \in \mathcal{L}_\Box$, and a class of frames/models $\mathfrak{S}$, φ is a logical consequence of Γ over $\mathfrak{S}$, denoted $\Gamma \models_{\mathfrak{S}} \varphi$, iff for all models $\mathcal{M}$ from $\mathfrak{S}$ and x in $\mathcal{M}$, if $\mathcal{M}, x \models \Gamma$ then $\mathcal{M}, x \models \varphi$.[1] It is not difficult to see that $\emptyset \models_{\mathfrak{S}} \varphi$ iff

[1]Following (Blackburn et al., 2001, Chapter 1.5), if $\mathfrak{S}$ is a class of models, then a model from $\mathfrak{S}$ is simply an element of $\mathfrak{S}$. If it is a class of frames, then a model from $\mathfrak{S}$ is a model based on a frame in $\mathfrak{S}$.

$\mathfrak{S} \models \varphi$, that is, validity in $\mathfrak{S}$ coincides with logical consequence from the empty set. We omit the subscript for the class of frames/models when it is clear from the context.

We define $\|\varphi\|^{\mathcal{M}} = \{x \in X \mid \mathcal{M}, x \models \varphi\}$ and call $\|\varphi\|^{\mathcal{M}}$ the *truth set*, or equivalently, *extension* of φ in $\mathcal{M}$. In particular, we write $x \in \|\varphi\|^{\mathcal{M}}$ for $\mathcal{M}, x \models \varphi$. We omit the superscript $\mathcal{M}$ when the model is clear from the context. The crucial concepts of *soundness* and *(strong) completeness* that link the syntax and the semantics are defined in the standard way (see, e.g., Blackburn et al., 2001, Chapter 4.1).

Definition 4 (Soundness). *Given $\Gamma \cup \{\varphi\} \subseteq \mathcal{L}_\Box$ and a class of frames/models $\mathfrak{S}$, a logic* L *is* (strongly) sound *with respect to $\mathfrak{S}$ iff if $\Gamma \vdash \varphi$, then $\Gamma \models_{\mathfrak{S}} \varphi$.*

Definition 5 (Strong Completeness). *Given $\Gamma \cup \{\varphi\} \subseteq \mathcal{L}_\Box$ and a class of frames/models $\mathfrak{S}$, a logic* L *is* strongly complete *with respect to $\mathfrak{S}$ iff if $\Gamma \models_{\mathfrak{S}} \varphi$, then $\Gamma \vdash \varphi$. It is* weakly complete *(or, simply* complete*) iff if $\mathfrak{S} \models \varphi$, then $\vdash \varphi$.*

We conclude the section by listing the relational soundness and completeness results for the important epistemic and doxastic logics defined in Table 2. To do so, we first list in Table 3 some important frame conditions, and then define some useful order theoretic notions that will also be used in later chapters.

Reflexivity	$(\forall x)(xRx)$
Transitivity	$(\forall x, y, z)(xRy \wedge yRz \to xRz)$
Symmetry	$(\forall x, y)(xRy \to yRx)$
Antisymmetry	$(\forall x, y)(xRy \wedge yRx \to x = y)$
Seriality	$(\forall x)(\exists y)(xRy)$
Euclideanness	$(\forall x, y, z)(xRy \wedge xRz \to yRz)$
Directedness [2]	$(\forall x, y, z)((xRy \wedge xRz) \to (\exists w)(yRw \wedge zRw))$
No right branching	$(\forall x, y, z)((xRy \wedge xRz) \to (yRz \vee zRy \vee y = z))$
Total (Connected)	$(\forall x, y)(xRy \vee yRx)$
Preorder	reflexive and transitive
Partial order	reflexive, transitive and antisymmetric
Equivalence relation	reflexive, transitive and symmetric

Table 3: Relevant Frame Conditions

Following the traditional conventions in order theory, we also call a reflexive and transitive relational frame (X, R) a *preordered set*; and a reflexive, transitive and antisymmetric frame a *partially ordered set*, or, in short, a *poset*. The following order theoretic notions will be useful in later chapters.

Definition 6 (Up/Down-set,Upward/Downward-closure). *Given a preordered set (X, R) and a subset $A \subseteq X$,*

- *A is called an* upward-closed set *(or, in short, an* up-set*) of (X, R) if for each $x, y \in X$, xRy and $x \in A$ imply $y \in A$;*

[2] Directedness is also called *confluence* or the *Church-Rosser property*.

- *A is called a* downward-closed set *(or, in short, a* down-set*) of* (X, R) *if for each* $x, y \in X$, yRx *and* $x \in A$ *imply* $y \in A$*;*

- *the* upward-closure *of* A*, denoted by* $\uparrow A$*, is the smallest up-set of* (X, R) *that includes* A*. In other words,* $\uparrow A = \{y \in X \mid \exists x \in A \text{ with } xRy\}$*;*

- *the* downward-closure *of* A*, denoted by* $\downarrow A$*, is the smallest down-set of* (X, R) *that includes* A*. In other words,* $\downarrow A = \{x \in X \mid \exists y \in A \text{ with } xRy\}$*.*

For every element $x \in X$*, we simply write* $\uparrow x$ *and* $\downarrow x$ *for the upward and downward-closure of the singleton* $\{x\}$*, respectively.*

We can now state some of the well-known relational soundness and strong completeness results. For a more detailed discussion, we refer to Chagrov & Zakharyaschev (1997); Blackburn et al. (2001).

Theorem 1 (Relational (Kripke) Completeness)**.**

- $\mathsf{S4}_\Box$ *is sound and strongly complete with respect to the class of preordered sets;*

- $\mathsf{S4.2}_\Box$ *is sound and strongly complete with respect to the class of directed preordered sets;*

- $\mathsf{S4.3}_\Box$ *is sound and strongly complete with respect to the class of total preordered sets;*

- $\mathsf{S5}_\Box$ *is sound and strongly complete with respect to the class of frames with equivalence relations;*

- $\mathsf{KD45}_\Box$ *is sound and strongly complete with respect to the class of serial, transitive and Euclidean frames.*

Following Theorem 1, we sometimes refer to a class of relational frames/models by the name of its corresponding logic. For example, a preordered set is also called an $\mathsf{S4}$-frame. Similarly, a relational model based on a serial, transitive and Euclidean frame is also called a $\mathsf{KD45}$-model, etc.

1.2 Background on Topology

In this section, we introduce the topological concepts that will be used throughout the course. We refer to (Dugundji, 1965; Engelking, 1989) for a thorough introduction to topology.

Definition 7 (Topological Space)**.** *A* topological space *is a pair* (X, τ)*, where* X *is a nonempty set and* τ *is a family of subsets of* X *such that*

1. $X, \emptyset \in \tau$*, and*

2. τ *is closed under finite intersections and arbitrary unions.*

The set X is a *space*; the family τ is called a *topology* on X. The elements of τ are called *open sets* (or *opens*) in the space. A set $C \subseteq X$ is called a *closed set* if it is the complement of an open set, i.e., it is of the form $X \setminus U$ for some $U \in \tau$. We let $\overline{\tau} = \{X \setminus U \mid U \in \tau\}$ denote the family of all closed sets of (X, τ). Moreover a set $A \subseteq X$ is called *clopen* if it is both closed and open.

Example 2. $\{\emptyset, X\}$ *is called the* trivial topology *on* X*. Moreover, the power set* $\mathcal{P}(X)$ *of* X *constitutes a topology on* X *and it is called the* discrete topology*.*

Example 3. *Let* $X = \{1, 2, 3, 4\}$ *and* $\tau = \{\emptyset, \{1\}, \{1, 2\}, \{3, 4\}, \{1, 3, 4\}, \{1, 2, 3, 4\}\}$*. It is easy to check that* τ *is closed under arbitrary unions and finite intersections, and* $\emptyset$ *and* X *are in* τ*, so* (X, τ) *is a topology.*
Note: *There was a mistake in the version distributed on July 9, 2025: the set* $\{1, 3, 4\}$ *was not included in the topology.*

Example 4. *A well-known example of a topology is on the real line. First, let* $B = \{(a, b) \mid a, b \in \mathbb{R}$ *and* $a < b\}$*, using standard interval notation:* (a, b) *denotes the set of all points strictly greater than* a *and strictly less than* b*. Then, for* $O \subseteq \mathbb{R}$*,* $O \in \tau$ *iff there exists some indexing set* I *such that* $O = \bigcup_{i \in I} b_i$ *where all* $b_i \in B$*. (Note that* I *may be uncountable.) Thus, open sets include* $(-1, 1)$*,* $(-2, 2) \cup (3, 4)$*, and* $(1, \infty)$*, since the last set is equal to* $\bigcup_{i=2}^{\infty}(1, i)$*.* $[1, 2]$ *and* $[1, 1]$ *are closed sets, while* $[10, 11)$ *is neither closed nor open. The set*

$$\bigcup_{i=0}^{\infty}(\frac{2^i - 1}{2^i}, \frac{2^{i+1} - 1}{2^{i+1}}) = (0, \frac{1}{2}) \cup (\frac{1}{2}, \frac{3}{4}) \cup (\frac{3}{4}, \frac{7}{8}) \cup \ldots$$

is an open set.

We leave it to the reader to verify that this topology satisfies the two required properties (1) and (2) in Definition 7. This topology is sometimes called the standard *(or,* natural*) topology* on the real line*. We call it the* standard topology *if the context is clear.*

Example 5. *Let* $(\mathbb{R}, \tau')$ *be the standard topology on the real line. Let* $X = [0, 1]$*, and* $\tau = \{O \mid O = O' \cap [0, 1]$ *for some* $O' \in \tau'\}$*. Then* (X, τ) *is a topology. This topology may be called* the standard *(or* natural*) topology on the unit interval.*

Example 6. *For* $S \subseteq \mathbb{N}$*, we say that* S *is* cofinite *if the complement of* S *is a finite set. (The complement of* S *is the set of all natural numbers that are not in* S*). For the space* $\mathbb{N}$*, the set of all cofinite sets and the empty set constitutes a topology on* $\mathbb{N}$*. This is called the* cofinite topology*.*

If for some $x \in X$ and an open $U \subseteq X$ we have $x \in U$, we say that U is an *open neighborhood* of x. A point x is called an *interior point* of a set $A \subseteq X$ if there is an open neighbourhood U of x such that $U \subseteq A$. The set of all interior points of A is called the *interior* of A and is denoted by $Int(A)$. Then, for any $A \subseteq X$, $Int(A)$ is an open set and is indeed the largest open subset of A, that is

$$Int(A) = \bigcup\{U \in \tau \mid U \subseteq A\}.$$

Dually, for any $x \in X$, x belongs to the *closure* of A, denoted by $Cl(A)$, if and only if $U \cap A \neq \emptyset$ for each open neighborhood U of x. It is not hard to see that $Cl(A)$ is the smallest closed set containing A, that is

$$Cl(A) = \bigcap\{C \in \overline{\tau} \mid A \subseteq C\},$$

and that $Cl(A) = X \setminus Int(X \setminus A)$ for all $A \subseteq X$. It is well known that the interior Int and the closure Cl operators of a topological space (X, τ) satisfy the following properties (the so-called Kuratowski axioms) for any $A, B \subseteq X$ (see, e.g., Engelking, 1989, pp. 14-15)[3]:

(I1) $Int(X) = X$	(C1) $Cl(\emptyset) = \emptyset$
(I2) $Int(A) \subseteq A$	(C2) $A \subseteq Cl(A)$
(I3) $Int(A \cap B) = Int(A) \cap Int(B)$	(C3) $Cl(A \cup B) = Cl(A) \cup Cl(B)$
(I4) $Int(Int(A)) = Int(A)$	(C4) $Cl(Cl(A)) = Cl(A)$

[3]The properties (I1) – (I4) (and, dually, (C1) – (C4)) are what render the knowledge modality interpreted as the topological interior operator an S4-type modality. We will elaborate on this in Section 2.

A set $A \subseteq X$ is called *dense* in X if $Cl(A) = X$ and it is called *nowhere dense* if $Int(Cl(A)) = \emptyset$. Moreover, the *boundary* of a set $A \subseteq X$, denoted by $Bd(A)$, is defined as $Bd(A) = Cl(A) \setminus Int(A)$.

A point $x \in X$ is called a *limit point* (or *accumulation point*) of a set $A \subseteq X$ if for each open neighborhood U of x, we have $A \cap (U \setminus \{x\}) \neq \emptyset$. The set of all limit points of A is called the *derived set* of A and is denoted by $d(A)$. For any $A \subseteq X$, we also let $t(A) = X \setminus d(X \setminus A)$. We call $t(A)$ the *co-derived set* of A. Moreover, a set $A \subseteq X$ is called *dense-in-itself* if $A \subseteq d(A)$. A space X is called *dense-in-itself* if $X = d(X)$.

Example 7. *In the standard topology on* $\mathbb{R}$,

- $Int([0, 1]) = (0, 1)$
- $Int([0, 1) \cup (1, 2]) = (0, 1) \cup (1, 2)$
- $Int((-1, -0.5)) = (-1, -0.5)$
- $Cl((9, 10) \cup (10, 11)) = [9, 11]$
- $Cl([2, 3)) = [2, 3]$
- $Cl(\bigcup_{i=0}^{\infty}(\frac{2^i-1}{2^i}, \frac{2^{i+1}-1}{2^{i+1}})) = [0, 1]$.

Example 8. *In the standard topology on* $\mathbb{R}$*, the set of rational numbers is dense.*

Proof. To show that the set of rational numbers is dense in $\mathbb{R}$, we must show that the closure of the set of rational numbers, $\mathbb{Q}$, is $\mathbb{R}$. This is easiest to prove by contradiction: suppose that there exists $x \in \mathbb{R}$ such that $x \notin Cl(\mathbb{Q})$. Then, since the closure of $\mathbb{Q}$ is the intersection of all closed sets containing $\mathbb{Q}$, there must be some closed set C such that $x \notin C$. Since C is closed, $\overline{C}$ is open, and $x \in \overline{C}$, but $\overline{C} \cap \mathbb{Q} = \emptyset$. However, every open set must contain some rational number, since an open set under the standard topology in $\mathbb{R}$ is a union of open intervals, and it is a well known proof that every open interval contains a rational number. Therefore, every open set contains a rational number, and we have a contradiction. □

Example 9. *For the space* $\mathbb{R}$ *with the standard topology,* $\mathbb{N}$ *is nowhere dense.*

Example 10. *For the standard topology on* $\mathbb{R}$*,* $Bd((0, 1)) = \{0, 1\}$*, and*

$$Bd(\bigcup_{i=0}^{\infty}(\frac{2^i-1}{2^i}, \frac{2^{i+1}-1}{2^{i+1}})) = \{0\} \cup \{\frac{2^i-1}{2^i}\} \mid i \in \mathbb{N}\}$$

Note: There was a typo in the version distributed on July 9, 2025: The second union "$\{\frac{2^i-1}{2^i}\}$" was wrongly written as "$\{\frac{1}{2^i}\}$".

Definition 8 (Topological Basis)**.** *A family* $\mathcal{B} \subseteq \tau$ *is called a* basis *for a topological space* (X, τ) *if every non-empty open subset of* X *can be written as a union of elements of* $\mathcal{B}$.

We call the elements of $\mathcal{B}$ *basic opens*. We can give an equivalent definition of an interior point by referring only to a basis $\mathcal{B}$ for a topological space (X, τ): for any $A \subseteq X$, $x \in Int(A)$ if and only if there is an open set $U \in \mathcal{B}$ such that $x \in U$ and $U \subseteq A$.

Example 11. *The set* $\{(a, b) \mid a, b \in \mathbb{R}$ *and* $a < b\}$ *is a basis for the standard topology on* $\mathbb{R}$.

Example 12. *The set* $\{(a, b) \mid a, b \in \mathbb{Q}$ *and* $a < b\}$ *is also a basis for the standard topology on* $\mathbb{R}$.

Example 13. *The set* $\{(a,b) \mid a,b \in \mathbb{Q}$ *and* $0 < a < b < 1\} \cup \{[0,b) \mid b \in \mathbb{Q}$ *and* $0 < b < 1\} \cup \{(a,1] \mid a \in \mathbb{Q}$ *and* $0 < a < 1\}$ *is a basis for the standard topology on* $[0,1]$.

Given any family $\Sigma = \{A_\alpha \mid \alpha \in I\}$ of subsets of X, there exists a unique, smallest topology $\tau(\Sigma)$ with $\Sigma \subseteq \tau(\Sigma)$ (Dugundji, 1965, Theorem 3.1, page 65). The family $\tau(\Sigma)$ consists of $\emptyset$, X, all finite intersections of the A_α, and all arbitrary unions of these finite intersections. Σ is called a *subbasis* for $\tau(\Sigma)$, and $\tau(\Sigma)$ is said to be *generated* by Σ. The set of finite intersections of members of Σ forms a basis for $\tau(\Sigma)$.

Example 14. *The set*

$$\{(-\infty, a) \mid a \in \mathbb{Q}\} \cup \{(b, \infty) \mid b \in \mathbb{Q}\}$$

is a subbasis for the standard topology on $\mathbb{R}$. *Is it also a basis?*

Example 15. *For* $X = \{0,1,2,3\}$*, let* $\Sigma = \{\{0,1\},\{1,2\},\{1,2,3\}\}$*. The topology generated by* Σ *is* $\{\emptyset,\{1\},\{0,1\},\{1,2\},\{0,1,2\},\{1,2,3\},\{0,1,2,3\}\}$.

Definition 9 (Subspace)**.** *Given a topological space* (X,τ) *and a nonempty subset* $P \subseteq X$*, the topological space* (P,τ_P) *is called a* subspace *of* (X,τ) *(*induced by P*) where* $\tau_P = \{U \cap P \mid U \in \tau\}$.

The closure Cl_P, the interior Int_P and the derived set d_P operators of the subspace (P,τ_P) can be defined in terms of the closure, interior, and derived set operators of (X,τ) as, for all $A \subseteq P$,

$$\begin{aligned} Cl_P(A) &= Cl(A) \cap P \\ Int_P(A) &= Int((X \setminus P) \cup A) \cap P \\ d_P(A) &= d(A) \cap P. \end{aligned}$$

Example 16. *We recall Example 5: if* $(\mathbb{R},\tau')$ *is the standard topology on the real line, let* $X = [0,1]$*, and* $\tau = \{O \mid O = O' \cap [0,1]$ *for some* $O' \in \tau'\}$*. Then* (X,τ) *is a subspace of* $(\mathbb{R},\tau')$.

Definition 10 (Hereditary Property)**.** *A property of a topological space is called* hereditary *if each subspace of the space possesses this property.*

Lemma 17. *For any two topological space* (X,τ) *and* (X,τ')*, if* $\tau \subseteq \tau'$ *then* $Int_\tau(A) \subseteq Int_{\tau'}(A)$ *for all* $A \subseteq X$.

Exercises:

1. Why is $(\{1,2,3\},\{\emptyset,\{1\},\{2\},\{1,3\},\{1,2,3\}\})$ not a topological space?
2. Show that the cofinite topology on $\mathbb{N}$ given in Example 6 defines a topological space.
3. Let $X = \mathbb{N}$ and $\tau = \{S \subseteq \mathbb{N} \mid S$ is a finite set$\}$. Is (X,τ) a topological space? Why or why not?
4. Can you think of a topology for the space $\mathbb{R} \times \mathbb{R}$? Can you prove that it is a topology?
5. In the topological space $(\{a,b,c\},\{\emptyset,\{a\},\{a,b\},\{a,b,c\}\})$, what is:
   - $Int(\{b\})$?
   - $Int(\{a,b\})$?
   - $Cl(\{a\})$?
   - $Cl(\{c\})$?

6. Prove that for any open set O in any topological space, $Int(O) = O$.

7. Prove that for any closed set C in any topological space, $Cl(C) = C$.

8. Find an example of open sets A and B in the standard topology on $\mathbb{R}$ where $Cl(A \cap B) \neq Cl(A) \cap Cl(B)$.

9. On $\mathbb{N}$, what is the topology generated by $\Sigma = \{[0, n] \mid n \in \mathbb{N}\}$?

10. Prove Lemma 17.

2 The Interior Semantics

This section gives an overview of the essential technical preliminaries of the interior semantics. The presentation of this section follows (van Benthem & Bezhanishvili, 2007, Section 2).

2.1 Syntax and Semantics

We work with the basic epistemic language $\mathcal{L}_K$ as given in Definition 1. Since we examine the interior semantics in an epistemic context, we prefer to use the modality $K\varphi$ (instead of $\Box\varphi$) that is read as "the agent knows φ (is true)". The dual modality $\hat{K}$ for *epistemic possibility* is defined as $\hat{K}\varphi := \neg K \neg \varphi$.

Definition 11 (Topological Model)**.** *A* topological model *(or, in short, a* topo-model*)* $\mathcal{X} = (X, \tau, V)$ *is a triple, where* (X, τ) *is a topological space and* $V : \mathsf{Prop} \to \mathcal{P}(X)$ *is a valuation function.*

Definition 12 (Interior Semantics for $\mathcal{L}_K$)**.** *Given a topo-model* $\mathcal{X}=(X, \tau, V)$ *and a state* $x \in X$, truth *of a formula in the langauge* $\mathcal{L}_K$ *is defined recursively as follows:*

$$\begin{array}{lll}
\mathcal{X}, x \models p & \textit{iff} & x \in V(p) \\
\mathcal{X}, x \models \neg\varphi & \textit{iff} & \textit{not } \mathcal{X}, x \models \varphi \\
\mathcal{X}, x \models \varphi \wedge \psi & \textit{iff} & \mathcal{X}, x \models \varphi \textit{ and } \mathcal{X}, x \models \psi \\
\mathcal{X}, x \models K\varphi & \textit{iff} & (\exists U \in \tau)(x \in U \textit{ and } \forall y \in U,\ \mathcal{X}, y \models \varphi)
\end{array}$$

It is useful to note the derived semantics for $\hat{K}\varphi$:

$$\mathcal{X}, x \models \hat{K}\varphi \quad \text{iff} \quad (\forall U \in \tau)(x \in U \text{ implies } \exists y \in U,\ \mathcal{M}, y \models \varphi)$$

Truth and *validity* of a formula φ of $\mathcal{L}_K$ are defined in the same way as for the relational semantics. We here apply similar notational conventions as we have set in Section 1.1. We let $[\![\varphi]\!]^{\mathcal{X}} = \{x \in X \mid \mathcal{X}, x \models \varphi\}$ denote the *truth set*, or equivalently, *extension* of a formula φ in topo-model $\mathcal{X}$. We emphasize the difference between $||\varphi||^{\mathcal{M}}$ and $[\![\varphi]\!]^{\mathcal{X}}$: while the former refers to the truth set in a relational model under the standard relational semantics (Definition 3), the latter is defined with respect to topo-models and the interior semantics (Definition 12). We again omit the superscript for the model when it is clear from the context.

The semantic clauses for K and $\hat{K}$ give us exactly the interior and the closure operators of the corresponding model. In other words, according to the interior semantics, we have

$$\begin{array}{lll}
[\![K\varphi]\!] & = & Int([\![\varphi]\!]) \\
[\![\hat{K}\varphi]\!] & = & Cl([\![\varphi]\!]).
\end{array}$$

2.2 Connection between relational and topological models

As well known, there is a one-to-one correspondence between the relational semantics and the interior semantics at the level of reflexive and transitive frames: every reflexive and transitive Kripke frame corresponds to an *Alexandroff space* (defined below). The class of reflexive and transitive frames therefore forms a subclass of all topological spaces. This connection not only helps us to see how the interior semantics and the relational semantics relate to each other and how the former extends the latter, but it also provides a method to prove topological completeness results by using the already established results for the relational counterpart.

Definition 13 (Alexandroff space). *A topological space* (X, τ) *is an* Alexandroff space *if* τ *is closed under arbitrary intersections, i.e.,* $\bigcap \mathcal{A} \in \tau$ *for any* $\mathcal{A} \subseteq \tau$*.*

A topo-model $\mathcal{X} = (X, \tau, V)$ is called an *Alexandroff model* if (X, τ) is an Alexandroff space. A very important feature of an Alexandroff space (X, τ) is that every point $x \in X$ has a smallest open neighbourhood. Given a reflexive and transitive Kripke frame (X, R), we can construct an Alexandroff space (X, τ_R) by defining τ_R to be the set of all up-sets of subsets of X. The up-set $R(x) = \uparrow x = \{y \in X \mid xRy\}$ forms the smallest open neighborhood containing the point x. It is then not hard to see that the set of all down-sets of (X, R) coincides with the set of all closed sets in (X, τ_R), and that for any $A \subseteq X$, we have $Cl_{\tau_R}(A) = \downarrow A$, where Cl_{τ_R} denotes the closure operator of (X, τ_R). Conversely, for every topological space (X, τ), we define a *specialization preorder* $\sqsubseteq_\tau$ on X by

$$x \sqsubseteq_\tau y \text{ iff } x \in Cl(\{y\}) \text{ iff } (\forall U \in \tau)(x \in U \text{ implies } y \in U).$$

$(X, \sqsubseteq_\tau)$ is therefore a reflexive and transitive Kripke frame, i.e., a preordered set. Moreover, we have that $R = \sqsubseteq_{\tau_R}$, and that $\tau = \tau_{\sqsubseteq_\tau}$ if and only if (X, τ) is Alexandroff (see, e.g., van Benthem & Bezhanishvili, 2007). Hence, there is a natural one-to-one correspondence between reflexive and transitive Kripke models and Alexandroff models. In particular, for any reflexive and transitive Kripke model $\mathcal{M} = (X, R, V)$, we set $B(\mathcal{M}) = (X, \tau_R, V)$, and for any Alexandroff model $\mathcal{X} = (X, \tau, V)$, we can form a reflexive and transitive Kripke model $A(\mathcal{X}) = (X, \sqsubseteq_\tau, V)$. Moreover, any two models that correspond to each other in the above mentioned way make the same formulas of $\mathcal{L}_K$ true at the same states, as shown in Proposition 18.[4]

Proposition 18. *For all* $\varphi \in \mathcal{L}_K$*,*

1. *for any reflexive and transitive Kripke model* $\mathcal{M} = (X, R, V)$ *and* $x \in X$*,*

$$\mathcal{M}, x \models \varphi \textit{ iff } B(\mathcal{M}), x \models \varphi;$$

2. *for any Alexandroff model* $\mathcal{X} = (X, \tau, V)$ *and* $x \in X$*,*

$$\mathcal{X}, x \models \varphi \textit{ iff } A(\mathcal{X}), x \models \varphi.$$

Therefore, reflexive and transitive Kripke models and Alexandroff models are just different representations of each other with respect to the language $\mathcal{L}_K$. In particular, the modal equivalence stated in Proposition 18-(1) constitutes the key step that allows us to use the relational completeness results to prove completeness with respect to the interior semantics.

[4]In this proposition, when there is a relational model on the left-hand side of $\models$, this sign represents the relational semantics given in Definition 3. When there is a topo-model, it represents the interior semantics given in Definition 12. We hope that this overlap of notation will not confuse the reader and relevant semantic clauses are contextually.

2.3 Soundness and Completeness for $\mathsf{S4}_K$, $\mathsf{S4.2}_K$ and $\mathsf{S4.3}_K$

Having explained the connection between reflexive-transitive Kripke models and Alexandroff models, we can now state the topological completeness results for $\mathsf{S4}_K$ and its two normal extensions $\mathsf{S4.2}_K$ and $\mathsf{S4.3}_K$ that are of interests in later sections. In fact, Proposition 18-(1) entails the following more general result regarding all Kripke complete normal extensions of $\mathsf{S4}_K$.

Proposition 19 (van Benthem & Bezhanishvili, 2007)**.** *Every normal extension of* $\mathsf{S4}_K$ *(over the language* $\mathcal{L}_K$*) that is complete with respect to the standard relational semantics is also complete with respect to the interior semantics.*

Proof. Let L_K be a normal extension of $\mathsf{S4}_K$ that is complete with respect to the relational semantics and $\varphi \in \mathcal{L}_K$ such that $\varphi \notin \mathsf{L}_K$. Then, by relational completeness of L_K, there exists a relational model $\mathcal{M} = (X, R, V)$ and $x \in X$ such that $\mathcal{M}, x \not\models \varphi$. Since L_K extends the system $\mathsf{S4}_K$, which is complete with respect to reflexive and transitive Kripke models, R can be assumed to be at least reflexive and transitive. Then, by Proposition 18-(1), we obtain $B(\mathcal{M}), x \not\models \varphi$. □

We can therefore prove completeness of the Kripke complete extensions of $\mathsf{S4}_K$ with respect to the interior semantics via their relational completeness. What makes the interior semantics more general than Kripke semantics is tied to soundness. For example, $\mathsf{S4}_K$ is not only sound with respect to Alexandroff spaces, but also with respect to all topological spaces.

Theorem 20 (McKinsey & Tarski, 1944)**.** $\mathsf{S4}_K$ *is sound and complete with respect to the class of all topological spaces under the interior semantics.*

Similar results have also been proven for $\mathsf{S4.2}_K$ and $\mathsf{S4.3}_K$ for the following restricted classes of topological spaces.

Definition 14 (Extremally Disconnected Space)**.** *A topological space* (X, τ) *is called* extremally disconnected *if the closure of each open subset of* X *is open.*

Example 21. *Alexandroff spaces constructed from directed preorders, i.e., from* $\mathsf{S4.2}_K$*-frames, are extremally disconnected. To elaborate, it is routine to verify that, given a directed preordered set* (X, R) *and an up-set* U *of* (X, R)*, the downward-closure* $\downarrow U$ *of the set* U *is still an up-set. Recall that* $Cl_{\tau_R}(U) = \downarrow U$*, where* (X, τ_R) *is the corresponding Alexandroff space and* Cl_{τ_R} *is its closure operator. Therefore, since the set of all up-sets of* (X, R) *forms the corresponding Alexandroff topology* τ_R*, we conclude that* (X, τ_R) *is extremally disconnected. This, in fact, establishes the topological completeness result for* $\mathsf{S4.2}_K$ *via Proposition 19. It is also well known that topological spaces that are Stone-dual to complete Boolean algebras, e.g., the Stone-Čech compactification* $\beta(\mathbb{N})$ *of the set of natural numbers with a discrete topology, are extremally disconnected (Sikorski, 1964).*

Definition 15 (Hereditarily Extremally Disconnected Space)**.** *A topological space* (X, τ) *is called* hereditarily extremally disconnected *(*h.e.d.*) if every subspace of* (X, τ) *is extremally disconnected.*

Example 22. *Alexandroff spaces corresponding to total preorders, i.e., corresponding to* $\mathsf{S4.3}_K$*-frames, are hereditarily extremally disconnected. To see this, observe that for every nonempty* $Y \subseteq X$*, the subspace* $(Y, (\tau_R)_Y)$ *of* (X, τ_R) *is in fact the Alexandroff space constructed from the subframe* $(Y, R \cap (Y \times Y))$ *of* (X, R)*. Moreover, every subframe of a total preorder* (X, R) *is still a total preorder, thus, is also a directed preorder. Therefore, the correspondence between total preorders and h.e.d spaces follows from the fact that Alexandroff spaces constructed from* *directed preorders are extremally disconnected. Another interesting and non-Alexandroff example of an hereditarily extremally disconnected space is the topological space $(\mathbb{N}, \tau)$ where $\mathbb{N}$ is the set of natural numbers and $\tau = \{\emptyset,$ all cofinite subsets of $\mathbb{N}\}$. In this space, the set of all finite subsets of $\mathbb{N}$ together with $\emptyset$ and X completely describes the set of closed subsets with respect to $(\mathbb{N}, \tau)$. It is not hard to see that for any $U \in \tau$, $Cl(U) = \mathbb{N}$ and $Int(C) = \emptyset$ for any closed C with $C \neq X$. Moreover, every countable Hausdorff extremally disconnected space is hereditarily extremally disconnected (Blaszczyk et al., 1993). For more examples of hereditarily extremally disconnected spaces, we refer to (Blaszczyk et al., 1993).*

Theorem 23 (Gabelaia, 2001)**.** *$\mathsf{S4.2}_K$ is sound and complete with respect to the class of extremally disconnected topological spaces under the interior semantics.*

Theorem 24 (Bezhanishvili et al., 2015)**.** *$\mathsf{S4.3}_K$ is sound and complete with respect to the class of hereditarily extremally disconnected topological spaces under the interior semantics.*

Exercises:

1. Prove Theorems 23 and 24 by using a similar reasoning as in Proposition 19. (Hint: Use Theorem 1 and consider the Alexandroff spaces given in Examples 21 and 22.)

Having presented the interior semantics, we elaborate on its epistemic significance in the following section.

2.4 The Motivation behind *Knowledge as Interior*

Note that the conception of *knowledge as interior* is not the only type of knowledge we study throughout this course. In fact, we favor the "knowability" interpretation of the interior operator (we will discuss this notion of knowability later, see, e.g., Bjorndahl (2018)). However, the aforementioned semantics can be considered as the most primitive, in a sense as the most direct way of interpreting an epistemic modality in this setting. Even in this very basic form, the interior semantics works at least as well as the standard relational semantics for knowledge, and, additionally, it extends the relational semantics while admitting an evidential interpretation of knowledge.

The interior semantics is naturally epistemic and extends the relational semantics. The initial reason as to why the topological interior operator can be considered as knowledge is inherent to the properties of this operator. As noted in Section 1.2, the Kuratowski axioms (I1)-(I4) correspond exactly to the axioms of the system $\mathsf{S4}_K$, when K is interpreted as the interior modality (see Table 4 for the one-to-one correspondence). Therefore, elementary topological

	$\mathsf{S4}_K$ axioms	Kuratowski axioms
(K_K)	$K(\varphi \wedge \psi) \leftrightarrow (K\varphi \wedge K\psi)$	$Int(A \cap B) = Int(A) \cap Int(B)$
(T_K)	$K\varphi \rightarrow \varphi$	$Int(A) \subseteq A$
(4_K)	$K\varphi \rightarrow KK\varphi$	$Int(A) \subseteq Int(Int(A))$
(Nec_K)	from φ, infer $K\varphi$	$Int(X) = X$

Table 4: $\mathsf{S4}_K$ vs. Kuratowski axioms

operators such as the interior operator, or, dually, the closure operator produces the epistemic logic $\mathsf{S4}_K$ with no need for additional constraints (also see Theorem 20). In other words, in its most general form, topologically modelled knowledge is *Factive* and *Positively Introspective*, however, it does not necessarily possess stronger properties. On the other hand, this in no way limits the usage of interior semantics for stronger epistemic systems. In accordance with the case for the relational semantics, we can restrict the class of spaces we work with and interpret stronger epistemic logics such as $\mathsf{S4.2}_K, \mathsf{S4.3}_K$ (see Theorems 23 and 24) and $\mathsf{S5}_K$ in a similar manner (see, e.g., van Benthem & Bezhanishvili, 2007, p. 253). To that end, topological spaces provide *sufficiently flexible* structures to study knowledge of different strength. They are moreover *naturally epistemic* since the most general class of spaces, namely the class of all topological spaces, constitutes the class of models of arguably the weakest, yet philosophically the most accepted normal system $\mathsf{S4}_K$. Moreover, as explained in Section 2.2, relational models for the logic $\mathsf{S4}_K$, and for its normal extensions, correspond to the subclass of Alexandroff models (see Proposition 18). The interior semantics therefore generalizes the standard relational semantics for knowledge.

One may however argue that the above reasons are more of a technical nature showing that the interior semantics works as well as the relational semantics, therefore motivate "why we *could* use topological spaces" rather than "why we *should* use topological spaces" to interpret knowledge, as opposed to using relational semantics. Certainly the most important argument in favour of the conception of *knowledge as the interior operator* is of a more 'semantic' nature: the interior semantics provides a deeper insight into the evidence-based interpretation of knowledge.

Evidence as open sets. The idea of treating 'open sets as pieces of evidence' is adopted from the topological semantics for intuitionistic logic, dating back to the 1930s (see, e.g., Troelstra & van Dalen, 1988). In a topological-epistemological framework, typically, the elements of a given open basis are interpreted as observable evidence, whereas the open sets of the topology are interpreted as properties that can be verified based on the observable evidence. In fact, the connection between evidence and open sets comes to exist at the most elementary level, namely at the level of a subbasis. We can think of a subbasis as a collection of observable evidence that is *directly* obtained by an agent via, e.g., testimony, measurement, approximation, computation or experiment (see, e.g., Özgün (2017); Baltag et al. (2022) for a more elaborate formalism of this interpretation. We will talk more about this throughout the course). The family of directly observable pieces of evidence therefore naturally forms an open topological basis: closure under finite intersection captures an agent's ability to put finitely many pieces into a single piece, i.e., her ability to derive more refined evidence from direct ones by combining finitely many of them together. Therefore, a topological space does not only account for the plain conception of *evidence as open sets*, but it is rich enough to differentiate various notions of evidence possession (again, see Özgün (2017); Baltag et al. (2022)). The above-mentioned correspondence between evidence and open sets constitutes the main motivation behind the topological frameworks we will examine throughout this course and we will elaborate on different views and interpretations of *topological evidence* as they come up.

On the other hand, the basic epistemic language $\mathcal{L}_K$ interpreted by the interior semantics is clearly not expressive enough to distinguish different types of open sets, e.g., it cannot distinguish a basic open from an arbitrary open, simply because the only topological modality K is interpreted as an existential claim of an open neighbourhood of the actual state that entails the known proposition:

$$\begin{aligned} x \in KP \text{ iff } & x \in Int(P) && (1)\\ \text{iff } & (\exists U \in \tau)(x \in U \text{ and } U \subseteq P) && (2)\\ \text{iff } & (\exists U \in \mathcal{B}_\tau)(x \in U \text{ and } U \subseteq P) && (3)\end{aligned}$$

where $\mathcal{B}_\tau$ is a basis for τ. Therefore, in its current form, the interior semantics does not form a sufficiently strong setting to account for (various type of) evidence possession alone. However, even based on this basic shape, the notion of knowledge as the interior operator yields an evidential interpretation at a purely semantic level. More precisely, from an extensional point of view[5], a proposition P is true at world x if $x \in P$. If an open U is included in a set P, then we can say that proposition P is *entailed/supported* by evidence U. Open neighbourhoods U of the actual world x play the role of sound (correct, truthful) evidence. Therefore, as basic open sets are the pieces of observable evidence, (3) means that the actual world x is in the interior of P iff there exists a sound piece of evidence U that supports P. That is, according to the interior semantics, the agent *knows* P at x iff she has a sound/correct piece of evidence supporting P. Moreover, open sets will then correspond to properties that are in principle verifiable by the agent: *whenever they are true, they are supported by a sound piece of evidence, therefore, can be known.* Dually, we have

$$x \notin Cl(P) \text{ iff } (\exists U \in \tau)(x \in U \text{ and } U \subseteq X \setminus P) \tag{4}$$

meaning that closed sets correspond to falsifiable properties: *whenever they are false, they are falsified by a sound piece of evidence.* These ideas have also been used and developed in (Vickers, 1989; Kelly, 1996) with connections to epistemology, logic and learning theory.

The interior-based semantics for knowledge has been extended to multiple agents (van Benthem et al., 2005), to common knowledge (Barwise, 1988; van Benthem & Sarenac, 2004) to logics of learning and observational *effort* (Moss & Parikh, 1992; Dabrowski et al., 1996; Georgatos, 1993, 1994), to topological versions of dynamic-epistemic logic (Zvesper, 2010) (see Aiello et al., 2007, for a comprehensive overview on the field).

In the second part of this course, we focus on the so-called Subset Space Logics for learning and observational *effort* introduced by Moss & Parikh (1992) and later further developed by Dabrowski et al. (1996); Georgatos (1993, 1994), together with their more contemporary doxastic extensions (see, e.g., Bjorndahl & Özgün 2017) and dynamic variants (see, e.g., Bjorndahl 2018; van Ditmarsch et al. 2019; Baltag et al. 2017, 2018, 2020).

3 Subset Space Semantics and TopoLogic

The formalism of "topologic", introduced by Moss & Parikh (1992), and investigated further by Dabrowski et al. (1996), Georgatos (1993, 1994), Weiss & Parikh (2002) and others, represents a *single-agent* subset space logic (SSL) for the notions of knowledge and *effort.* One of the crucial aspects of this framework is that it is concerned not only with the representation of knowledge, but also aimed at giving an account of information gain or knowledge increase in terms of *observational effort.*[6] It is the latter feature of this work that makes the use of subset spaces significant. While the knowledge modality $K\varphi$ has the standard reading "the agent knows φ (is true)", in the subset space setting, the effort modality $\Box\varphi$ captures a notion of effort as any action that results in an increase in knowledge and is read as "φ stays true no matter what further evidence-gathering efforts are made"[7]. The modality $\Box$ therefore captures a notion of *stability* under evidence-gathering. Effort can be in the form of measurement, computation, approximation, or even announcement, depending on the context and the information source. To

[5] *Extensional* here means any semantic formalism that assigns the same meaning to sentences having the same extension.

[6] Moss & Parikh (1992) is partly inspired by Vickers' work on reconstruction of topology via a logic of finite observation (Vickers, 1989).

[7] Please note the meaning change for $\Box$ in this framework. It is not the same modality as in Section 1.

illustrate the underlying intuition of the subset space semantics, and the notions of knowledge, effort, and evidence it represents, suppose for instance, that you have measured your height and obtained a reading of 5 feet and 10 inches ±3 inches. The measuring devices we use to calculate such quantities always come with an error range, therefore giving us an approximation rather than the precise value. With this measurement in hand, you cannot be said to know whether you are less than 6 feet tall, as your measurement, i.e., the current evidence you have, does not rule out that you are taller or shorter. However, if you are able to spend more resources and take a more precise measurement, e.g., by using a more accurate meter with ± 1 error range, you come to know that you are less than 6 feet tall (Bjorndahl & Özgün, 2017). Subset space logics are designed to represent such situations, and therefore involve two modalities: one for knowledge K, and the other one for effort $\Box$.

The formulas in the bimodal language are interpreted on subset spaces $(X, \mathcal{O})$, where X is a nonempty domain and $\mathcal{O}$ is an *arbitrary* nonempty collection of subsets of X. The elements of $\mathcal{O}$ represent *possible observations*, and more effort corresponds to a more refined truthful observation, thus, a possible increase in knowledge. A subset space is not necessarily a topological space, however, topological spaces do constitute a subclass of subset spaces and topological reasoning provides the intuition behind this semantics, as we will elaborate below.[8]

In this section, we provide the formal background for the subset space semantics of Moss & Parikh (1992), explaining how these "topological" structures constitute models that are well-equipped to give an account for evidence-based knowledge and its dynamics. We also point out the differences and connection to the topological approaches we presented earlier in the course. In particular, we compare the evidence representation on evidence models of van Benthem & Pacuit (2011) with the one on subset models of Moss & Parikh (1992), and in turn, the type of evidence-based knowledge studied on these structures.

3.1 Syntax and Semantics

In their influential work, Moss & Parikh (1992) consider the bimodal language $\mathcal{L}_{K\Box}$ given by the grammar

$$\varphi ::= p \mid \neg\varphi \mid \varphi \wedge \varphi \mid K\varphi \mid \Box\varphi,$$

and interpret it on subset spaces, a class of models generalizing topological spaces.

Definition 16 (Subset Space/Model). *A* subset space *is a pair $(X, \mathcal{O})$, where X is a nonempty set of states and $\mathcal{O}$ is a collection of subsets of X. A* subset model *is a tuple $\mathcal{X} = (X, \mathcal{O}, V)$, where $(X, \mathcal{O})$ is a subset space and $V : \mathsf{Prop} \to \mathcal{P}(X)$ a valuation function.*

It is not hard to see that subset spaces are just like the evidence models of van Benthem & Pacuit (2011) (see Slides), but with no constraints on the set of subsets $\mathcal{O}$.[9] However, the way the truth of a formula is defined on subset models leads to a crucial difference between the two settings, especially concerning the type of evidence represented by the elements of $\mathcal{O}$, and

[8] The subset space setting also comes with an independent technical motivation. Many of the aforementioned sources are concerned with axiomatizing the logics of smaller classes of subset spaces meeting particular closure conditions on the set of subsets $\mathcal{O}$. For example, while Moss & Parikh (1992) axiomatized the logic of subset spaces, Georgatos (1993, 1994) and Dabrowski et al. (1996) provided an axiomatization of the logic of topological spaces, and complete lattice spaces. Moreover, Georgatos (1997) axiomatized the logic of treelike spaces, and Weiss & Parikh (2002) presented an axiomatization for the class of directed spaces. These results are quite interesting from a modal theoretical perspective, however, in this course, we are primarily interested in the applications of topological ideas in epistemic logic. We therefore focus on the epistemic motivation behind the topologic formalism.

[9] We could in fact define the subset spaces exactly the same way as evidence models by putting the constraints $X \in \mathcal{O}$ and $\emptyset \notin \mathcal{O}$. This would technically make no difference, however, we here prefer to present the most general case.

the characterization of the notion of knowledge interpreted based on evidence. This point will become clear once we present the formal semantics below.

Subset space semantics interprets formulas not at worlds x but at *epistemic scenarios* of the form (x, U), where $x \in U \in \mathcal{O}$. Let $ES(\mathcal{X})$ denote the collection of all such pairs in $\mathcal{X}$. Given an epistemic scenario $(x, U) \in ES(\mathcal{X})$, the set U is called its *epistemic range*; intuitively, it represents the agent's current information as determined, for example, by the measurements she has taken. The language $\mathcal{L}_{K\Box}$ is interpreted on subset spaces as follows:

Definition 17 (Subset Space Semantics for $\mathcal{L}_{K\Box}$)**.** *Given a subset space model $\mathcal{X} = (X, \mathcal{O}, V)$ and an epistemic scenario $(x, U) \in ES(\mathcal{X})$,* truth *of a formula in the language $\mathcal{L}_{K\Box}$ is defined recursively as follows:*

$$\begin{array}{lll} \mathcal{X},(x,U) \models p & \textit{iff} & x \in V(p), \ \textit{where } p \in \mathsf{Prop} \\ \mathcal{X},(x,U) \models \neg\varphi & \textit{iff} & \textit{not } \mathcal{X},(x,U) \models \varphi \\ \mathcal{X},(x,U) \models \varphi \wedge \psi & \textit{iff} & \mathcal{X},(x,U) \models \varphi \textit{ and } \mathcal{X},(x,U) \models \psi \\ \mathcal{X},(x,U) \models K\varphi & \textit{iff} & (\forall y \in U)(\mathcal{X},(y,U) \models \varphi) \\ \mathcal{X},(x,U) \models \Box\varphi & \textit{iff} & (\forall O \in \mathcal{O})(x \in O \subseteq U \textit{ implies } \mathcal{X},(x,O) \models \varphi) \end{array}$$

We say that a formula φ is *valid in a model* $\mathcal{X}$, and write $\mathcal{X} \models \varphi$, if $\mathcal{X},(x,U) \models \varphi$ for all scenarios $(x,U) \in ES(\mathcal{X})$. We say φ is *valid*, and write $\models \varphi$, if $\mathcal{X} \models \varphi$ for all $\mathcal{X}$. We let $[\![\varphi]\!]^U_{\mathcal{X}} = \{x \in U \mid \mathcal{X},(x,U) \models \varphi\}$ denote the *truth set*, or equivalently, *extension of φ under U in the model $\mathcal{X}$*. We again omit the notation for the model, writing simply $(x,U) \models \varphi$ and $[\![\varphi]\!]^U$, whenever $\mathcal{X}$ is fixed.

3.2 Epistemic readings of subset space semantics: current vs potential evidence

In subset space semantics, the points of the space represent "possible worlds" (or, *states* of the world). However, having the units of evaluation as pairs of the form (x,U)—rather than a single state x—allows us to distinguish the evidence that the agent currently has in hand from the potential evidence she can *in principle* obtain. More precisely, elements of $\mathcal{O}$ can be thought of as potential pieces of evidence meant to encompass all the evidence that might be learnt in the future, while the epistemic range U of an epistemic scenario (x,U) corresponds to the current evidence, i.e., "evidence-in-hand" by means of which the agent's knowledge is evaluated.[10] This is made precise in the semantic clause for $K\varphi$, which stipulates that the agent knows φ just in case φ is entailed by her *factive*[11] evidence-in-hand. The knowledge modality K therefore behaves like the global modality within the given epistemic range U. For this reason, in various places, we will often refer to K as the global modality. Thus, the type of knowledge captured by the modality K in this setting is absolutely certain, infallible knowledge based on the agent's current truthful evidence. These points already underline the substantial differences between the two evidence-based epistemic frameworks we study throughout the course: while $\mathcal{E}_0$ of an evidence model $(X, \mathcal{E}_0, V)$ represents the set of evidence pieces the agent *has already acquired about the actual situation*, the set $\mathcal{O}$ of a subset space model $(X, \mathcal{O}, V)$ represents the set of *potential evidence the agent can in principle discover, even if she does not happen to personally have it in hand at the moment*. A subset space model is therefore intended to carry all pieces of evidence the agent currently has and can potentially gather later, hence, supports model-internal means to interpret evidence-based information dynamics, as displayed, e.g., by the effort modality. In this

[10] The term "evidence-in-hand" is borrowed from (Bjorndahl & Özgün, 2017), where the elements of $\mathcal{O}$ are described as "evidence-out-there".

[11] $x \in U$ expresses the factivity of evidence.

framework, more effort means acquiring more evidence for the actual state of affairs, therefore, a better approximation of the real state. The effort modality $\Box\varphi$ is thus interpreted in terms of *neighbourhood-shrinking* and read as "φ is stably true under evidence-acquisition", i.e., φ is true, and will stay true no matter what further factive evidence is obtained.

As every topological space is a subset space, the above readings of the modalities also apply to the topological models. However, the additional structure that topological spaces possess helps us to formalize naturally some further aspects of evidence aggregation. For example, when $\mathcal{O}$ is closed under finite intersections, we can consider the epistemic range U of a given epistemic scenario (x, U) as a finite stream $(O_1, \ldots, O_n)$ of truthful information the agent has received and put together: $x \in U = \bigcap_{i \leq n} O_i \in \mathcal{O}$ (Baltag et al., 2015, 2018). Moreover, as noted in (Moss & Parikh, 1992), we can express some topological concepts in the language $\mathcal{L}_{K\Box}$ that, in fact, lead to concise modal reformulations of *verifiable* and *falsifiable* propositions (as also noted in Georgatos, 1993). To be more precise, given a topo-model $\mathcal{X} = (X, \tau, V)$ and a propositional variable $p \in \mathsf{Prop}$, $V(p)$ is open in τ iff $p \to \Diamond Kp$ is valid in $\mathcal{X}$. Recall that the open sets of a topology are meant to represent potential evidence, i.e., properties of the actual state that are in principle *verifiable*: *whenever they are true, they are supported by a sound piece of evidence that the agent can in principle obtain, therefore, can be known* (Vickers, 1989; Kelly, 1996). Therefore, we can state that

- p is *verifiable* in $\mathcal{X}$ iff $p \to \Diamond Kp$ is valid in $\mathcal{X}$.

In contrast, $V(p)$ is closed in τ iff $\Box\hat{K}p \to p$ is valid in $\mathcal{X}$, and closed sets correspond to properties that are in principle *falsifiable*: *whenever they are false, their falsity can be known.* In a similar manner, this can be formalized in the language $\mathcal{L}_{K\Box}$ as

- p is *falsifiable* in $\mathcal{X}$ iff $\neg p \to \Diamond K\neg p$, or equivalently, $\Box\hat{K}p \to p$ is valid in $\mathcal{X}$.

As remarked in (Vickers, 1989; Kelly, 1996), the closure properties of a topology are satisfied in this interpretation. First, contradictions ($\emptyset$) and tautologies (X) are in principle verifiable (as well as falsifiable). The conjunction $p \wedge q$ of two verifiable facts is also verifiable: if $p \wedge q$ is true, then both p and q are true, and since both are assumed to be verifiable, they can both be known, and hence $p \wedge q$ can be known. Finally, if $\{p_i \mid i \in I\}$ is a (possibly infinite) family of verifiable facts, then their disjunction $\bigvee_{i \in I} p_i$ is verifiable: in order for the disjunction to be true, then there must exist some $i \in I$ such that p_i is true, and so p_i can be known (since it is verifiable), and as a result the disjunction $\bigvee_{i \in I} p_i$ can also be known (by inference from p_i).

Exercises:

Given a subset space model $(X, \mathcal{O}, V)$ and $p \in \mathsf{Prop}$, prove that

1. $V(p)$ is open in τ iff $p \to \Diamond Kp$ is valid in $\mathcal{X}$.
2. $V(p)$ is closed in τ iff $\Box\hat{K}p \to p$ is valid in $\mathcal{X}$.

3.3 Axiomatizations: SSL and TopoLogic

Moss & Parikh (1992) provided a sound and complete axiomatization of their logic of knowledge and effort with respect to the class of subset spaces. Its purely topological version was later studied by Georgatos (1993, 1994), and Dabrowski et al. (1996), who independently provided complete axiomatizations and proved decidability. In this section, we give the axiomatizations for the logic of subset spaces (SSL) and of topological spaces (TopoLogic). We state the relevant completeness, decidability and finite model property results, and refer to the aforementioned sources for their proofs.

The axiomatization of the subset space logic, denoted by SSL, is obtained by augmenting the logic $\mathsf{S5}_K + \mathsf{S4}_\Box$ for the language $\mathcal{L}_{K\Box}$ with the additional axiom schemes (AP) and (CA) presented in Table 5.

(AP)	$(p \to \Box p) \wedge (\neg p \to \Box \neg p)$, for $p \in \mathsf{Prop}$	Atomic Permanence
(CA)	$K\Box\varphi \to \Box K\varphi$	Cross Axiom

Table 5: Additional axiom schemes of SSL

Therefore, the effort modality on subset spaces is S4-like. The axiom (AP) states that the truth value of the propositional variables does not depend on the given epistemic range, but only depends on the actual state. In fact, this is the case for all Boolean formulas in $\mathcal{L}_{K\Box}$, and can be proven in the system SSL. The cross axiom is also interesting since it links the two modalities of this system.

Theorem 25 (Moss & Parikh, 1992)**.** SSL *is sound and complete with respect to the class of all subset spaces.*

Exercises:

1. Show that SSL is sound with respect to the class of all subset spaces.
2. Show that $\neg Kp \to \Box\neg Kp$ is not valid. What does this tell us about the axiom (AP)?

It was shown in (Dabrowski et al., 1996) that the logic of subset spaces does not have the finite model property, however, its decidability was proven by using non-standard models called *cross axiom* models (see Dabrowski et al., 1996, Section 2.3).

Concerning the logic of topological spaces for $\mathcal{L}_{K\Box}$, i.e., the so-called TopoLogic, it is axiomatized by adding the following axiom schemes to the axiomatization of SSL:

(WD)	$\Diamond\Box\varphi \to \Box\Diamond\varphi$	Weak Directedness
(UN)	$(\Diamond\varphi \wedge \hat{K}\Diamond\psi) \to \Diamond(\Diamond\varphi \wedge \hat{K}\Diamond\psi \wedge K\Diamond\hat{K}(\varphi \vee \psi))$	Union Axiom

Table 6: Additional axiom schemes of TopoLogic

Theorem 26 (Georgatos, 1993, 1994)**.** TopoLogic *is sound and complete with respect to the class of all topological spaces. Moreover, it has the finite model property, therefore, it is decidable.*

The literature on subset space semantics goes far beyond the presentation of this section. However, we here confine ourselves to the material we will present in our lectures, and refer to (Parikh et al., 2007) for a survey of the further technical results, extensions, and variations of the topologic formalism. Further extensions involve the dynamics of evidence acquisition: the connection between the effort modality, and the well-known dynamic epistemic modalities such as the public and arbitrary announcement modalities have been studied, e.g., in Bjorndahl (2018); van Ditmarsch et al. (2019); Baltag et al. (2017, 2018, 2020).

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