

LOGIC AND LEARNING

DAY 4: TRUTH-TRACKING AND MODAL LOGIC

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WHAT DO WE MEAN BY ‘LEARNING’?

General **qualitative** model of (exact) learning:

- ▶ on the basis of incoming data consistent with an underlying concept
- ▶ learner achieves a **desired type of knowledge** of the underlying concept.

This perspective in various ways generalises many popular learning topics:

- ▶ one step updates with an incoming piece of information:
Belief Revision Theory, Dynamic Epistemic Logic
- ▶ particular algorithmic probabilistic methods of automatic improvement:
Machine Learning, Bayesian Learning, Reinforcement Learning



Gierasimczuk, N., Learning by Erasing in Dynamic Epistemic Logic. LATA 2009.



Gierasimczuk, N., Bridging Learning Theory and Dynamic Epistemic Logic. Synthese 2009.



Gierasimczuk, N., Knowing One's Limits. Logical Analysis of Inductive Inference. PhD thesis, Universiteit van Amsterdam 2010.

OUTLINE

SUBSET SPACES, LEARNABILITY, AND SOLVABILITY

TOPO-CHARACTERIZATIONS OF LEARNABILITY AND SOLVABILITY

LEARNING AND MODAL LOGIC: THERE

OUTLINE

SUBSET SPACES, LEARNABILITY, AND SOLVABILITY

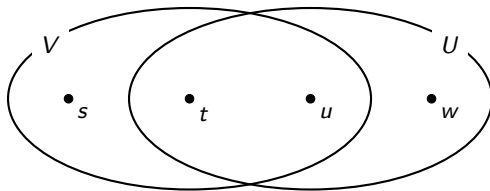
TOPO-CHARACTERIZATIONS OF LEARNABILITY AND SOLVABILITY

LEARNING AND MODAL LOGIC: THERE

SUBSET SPACE

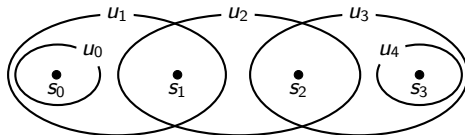
DEFINITION

A **subset space** is $(X, \mathcal{O})$, where $\mathcal{O} \subseteq \mathcal{P}(X)$, X and $\mathcal{O}$ (at most) countable.



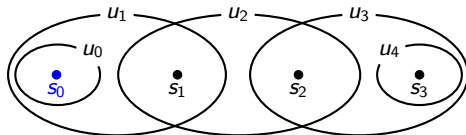
EXAMPLE: FINITE IDENTIFIABILITY

RESULTING KNOWLEDGE: CERTAINTY



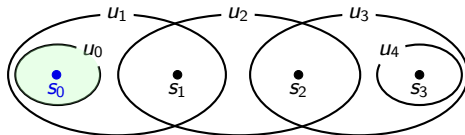
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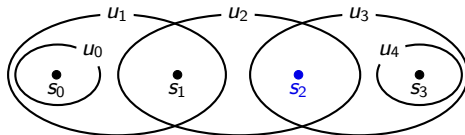
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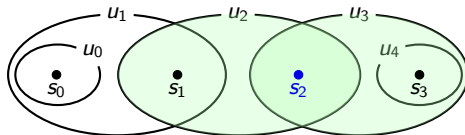
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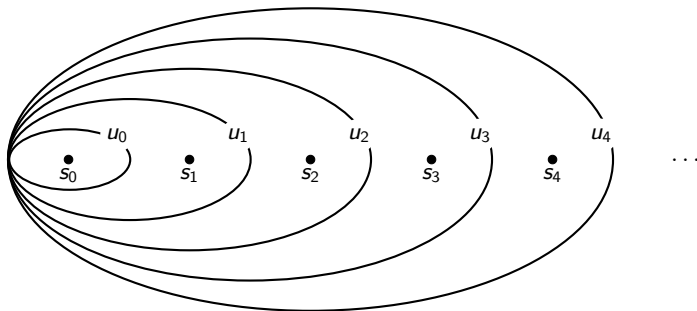
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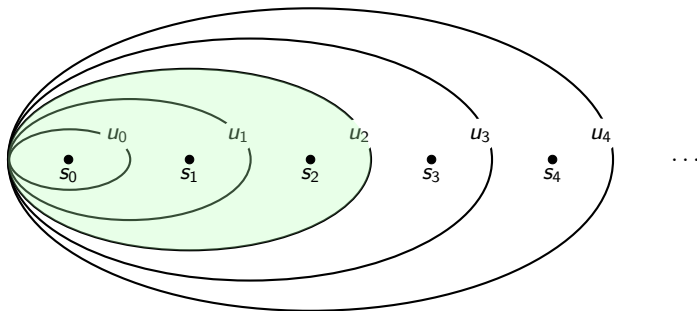
EXAMPLE: IDENTIFIABILITY IN THE LIMIT

RESULTING KNOWLEDGE: UNDEFEATED BELIEF



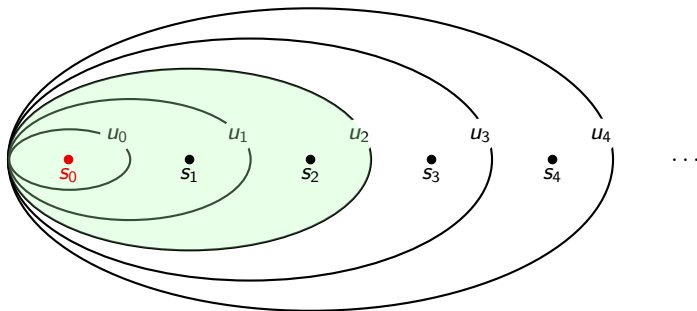
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RESULTING KNOWLEDGE: UNDEFEATED BELIEF



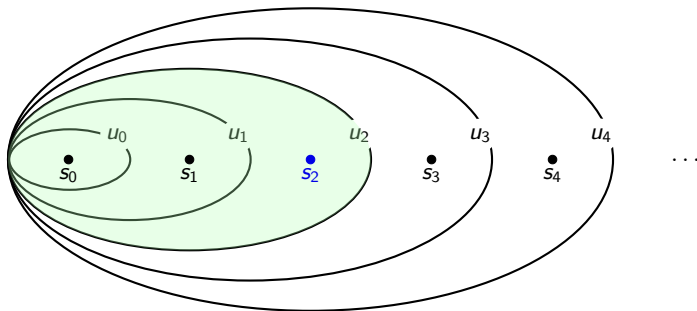
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RESULTING KNOWLEDGE: UNDEFEATED BELIEF

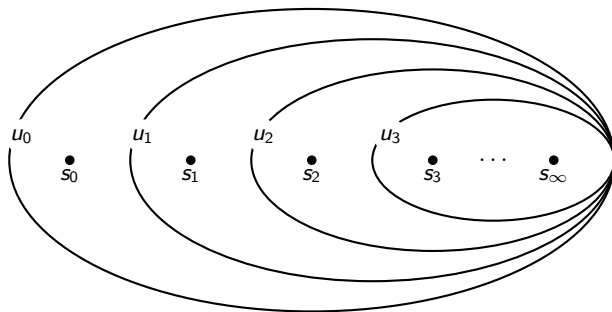


EXAMPLE: IDENTIFIABILITY IN THE LIMIT

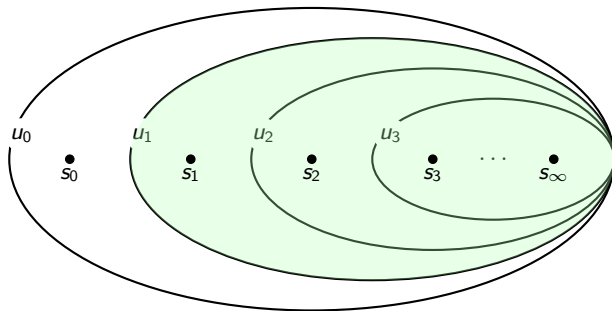
RESULTING KNOWLEDGE: UNDEFEATED BELIEF



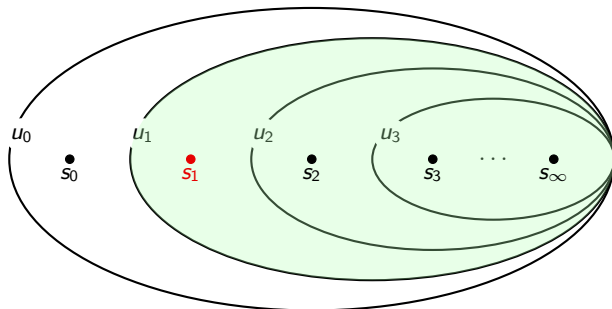
EXAMPLE: NON-IDENTIFIABILITY IN THE LIMIT



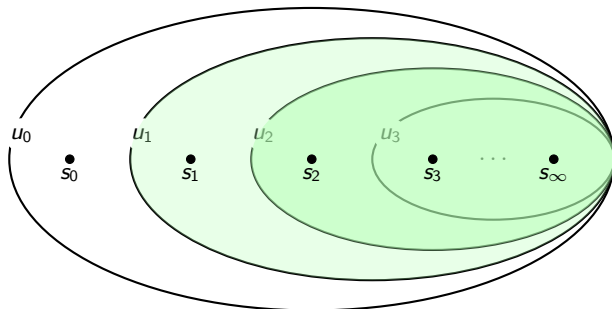
EXAMPLE: NON-IDENTIFIABILITY IN THE LIMIT



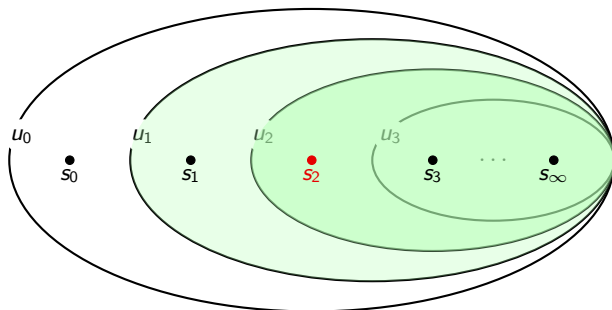
EXAMPLE: NON-IDENTIFIABILITY IN THE LIMIT



EXAMPLE: NON-IDENTIFIABILITY IN THE LIMIT



EXAMPLE: NON-IDENTIFIABILITY IN THE LIMIT



LEARNING: STREAMS OF OBSERVATIONS

DEFINITION

Let $(X, \mathcal{O})$ be a subset space.

- ▶ A **data stream** is an infinite sequence $\vec{O} = (O_0, O_1, \dots)$ from $\mathcal{O}$.
- ▶ A **data sequence** $\vec{O}[n]$ is a finite initial segment of $\vec{O}$ of length $n + 1$.

DEFINITION

Take $(X, \mathcal{O})$ and $s \in S$. A data stream $\vec{O}$ is:

- ▶ **sound with respect to** s iff every element listed in $\vec{O}$ is true in s .
- ▶ **complete with respect to** s iff every observable true in s is listed in $\vec{O}$.

We assume that data streams are sound and complete.

LEARNING: LEARNERS AND CONJECTURES

DEFINITION

Let $(X, \mathcal{O})$ be a subset space and let σ be a data sequence.

A **learner** L is a function that on σ outputs a conjecture $L(\sigma) \subseteq X$.

DEFINITION

$(X, \mathcal{O})$ is **identified in the limit by** L if for every $x \in X$ and every data stream $\vec{O}$ for x , there is $k \in \mathbb{N}$ s.t.:

$$L(\vec{O}[n]) = \{x\} \text{ for all } n \geq k.$$

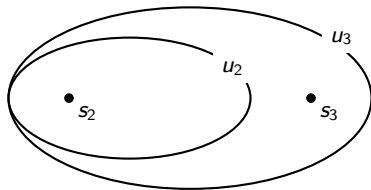
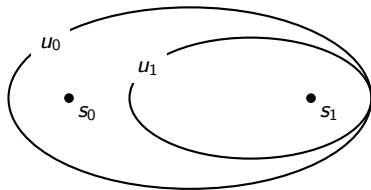
$(X, \mathcal{O})$ is **identifiable in the limit** if it is identified in the limit by a learner L .

QUESTIONS, ANSWERS, AND PROBLEMS

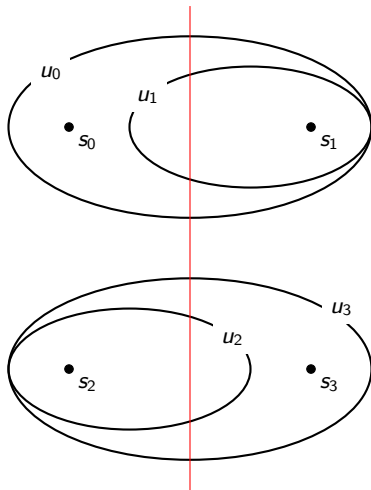
DEFINITION

- ▶ A **question** $\mathcal{Q}$ is a partition of X , whose cells A_i are called **answers to** $\mathcal{Q}$.
- ▶ Given $x \in A \subseteq X$, $A \in \mathcal{Q}$ is called **the answer to** $\mathcal{Q}$ **at** x , denoted A_x .
- ▶ $\mathcal{Q}'$ is a **refinement** of $\mathcal{Q}$ if answers of $\mathcal{Q}$ are disjoint unions of those of $\mathcal{Q}'$.
- ▶ A **problem** is a pair $((X, \mathcal{O}), \mathcal{Q})$, where $\mathcal{Q}$ is a question over X .
- ▶ $((X, \mathcal{O}), \mathcal{Q}')$ is a **refinement** of $((X, \mathcal{O}), \mathcal{Q})$ if $\mathcal{Q}'$ is a refinement of $\mathcal{Q}$.

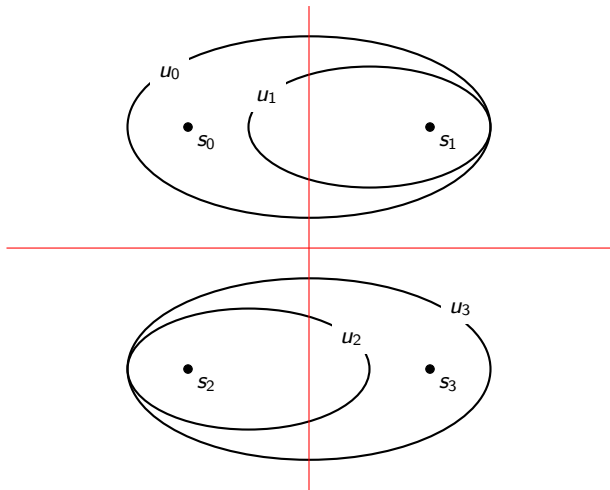
EXAMPLE: REFINEMENTS



EXAMPLE: REFINEMENTS



EXAMPLE: REFINEMENTS



SOLVING IN THE LIMIT

DEFINITION

$((X, \mathcal{O}), \mathcal{Q})$ is **solved in the limit by** L if for every $x \in X$ and every data stream $\vec{O}$ for x , there is $k \in \mathbb{N}$ s.t.:

$$L(\vec{O}[n]) \subseteq A_x \text{ for all } n \geq k.$$

$((X, \mathcal{O}), \mathcal{Q})$ is **solvable in the limit** if solved in the limit by a learner L .

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SUBSET SPACES, LEARNABILITY, AND SOLVABILITY

TOPO-CHARACTERIZATIONS OF LEARNABILITY AND SOLVABILITY

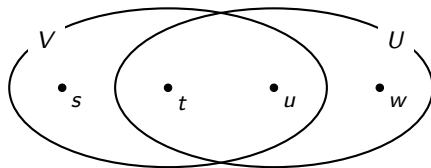
LEARNING AND MODAL LOGIC: THERE

GENERAL TOPOLOGY

DEFINITION

A subset space $(X, \mathcal{O})$ is topological if:

1. $\emptyset \in \mathcal{O}$,
2. $X \in \mathcal{O}$,
3. for any $Y \subseteq \mathcal{O}$, $\bigcup Y \in \mathcal{O}$, and
4. for any finite $Y \subseteq \mathcal{O}$, we have $\bigcap Y \in \mathcal{O}$.

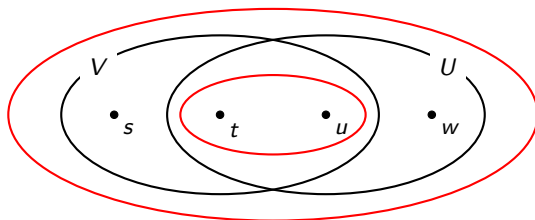


GENERAL TOPOLOGY

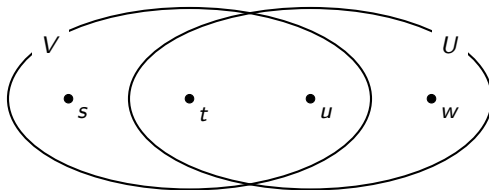
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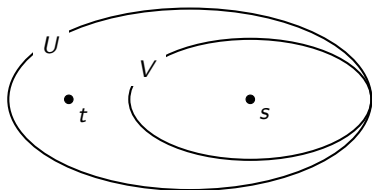
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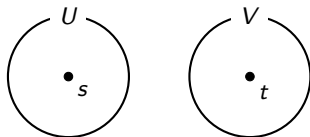
SEPARABILITY BY OBSERVATIONS: ILLUSTRATION



(A) t and u are not separable



(B) weakly separated space, $T0$



(C) strongly separated space, $T1$

LOCALLY CLOSED AND CONSTRUCTIBLE SETS

DEFINITION

A topological space $(X, \mathcal{O})$ is T_d iff
for every $x \in X$ there is a $U \in \mathcal{O}$ such that $U \setminus \{x\} \in \mathcal{O}$.

T_d is a separation property between T_0 and T_1 .

DEFINITION

A set A is **locally closed** if $A = U \cap C$, where U is open and C is closed.

CHARACTERIZATION OF SOLVABILITY IN THE LIMIT

THEOREM

$((X, \mathcal{O}), \mathcal{Q})$ is solvable in the limit iff $\mathcal{Q}$ has a locally closed refinement.

COROLLARY

$(X, \mathcal{O})$ is identifiable in the limit iff it is T_d .



A. Baltag, N. Gierasimczuk, S. Smets, On the solvability of inductive problems: a study in epistemic topology, TARK 2015.

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RELATIONAL SEMANTICS FOR MODAL LOGIC

DEFINITION (SYNTAX)

Let P be a countable set of propositional symbols, $p \in P$.

$$\varphi := p \mid \neg\varphi \mid \varphi \wedge \varphi \mid \Box\varphi$$

DEFINITION (SEMANTICS)

Given a model $M = (W, R, v)$, where $R \subseteq W \times W$, $v : P \rightarrow \wp(W)$, $x \in W$:

$M, x \models p$	iff	$x \in v(p)$ for each $p \in P$
$M, x \models \neg\varphi$	iff	not $M, x \models \varphi$
$M, x \models \varphi \wedge \psi$	iff	$M, x \models \varphi$ and $M, x \models \psi$
$M, x \models \Box\varphi$	iff	for all $y \in W$: if xRy then $M, y \models \varphi$

SOME AXIOMS AND THEIR EPISTEMIC INTERPRETATION

Rules

(MP) if $\vdash \varphi$ and $\vdash \varphi \rightarrow \psi$, then $\vdash \psi$

(N) if $\vdash \varphi$, then $\vdash \Box \varphi$

Axioms

(K) $\Box(\varphi \rightarrow \psi) \rightarrow (\Box \varphi \rightarrow \Box \psi)$ (omniscience)

(T) $\Box \varphi \rightarrow \varphi$ (truthfulness/reflexivity)

(D) $\Box \varphi \rightarrow \neg \Box \neg \varphi$ (consistency/seriality)

(4) $\Box \varphi \rightarrow \Box \Box \varphi$ (positive introspection/transitivity)

(5) $\neg \Box \varphi \rightarrow \Box \neg \Box \varphi$ (negative introspection/Euclidean-ness)

SOME AXIOMS AND THEIR EPISTEMIC INTERPRETATION

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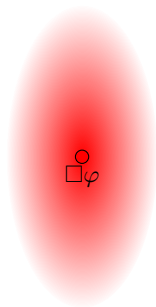
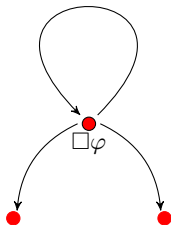
(4) $\Box\varphi \rightarrow \Box\Box\varphi$ (positive introspection/transitivity)

(5) $\neg\Box\varphi \rightarrow \Box\neg\Box\varphi$ (negative introspection/Euclidean-ness)

Ax is a logic of a class of models $\mathcal{M}$ iff Ax is sound and complete wrt $\mathcal{M}$.

TOPOLOGICAL INTERPRETATIONS

RELATIONAL $\square$ VS TOPOLOGICAL $\square := \textit{Int}$



TOPOLOGICAL TOPO-SEMANTICS FOR MODAL LOGIC

DEFINITION (SYNTAX)

Let P be a countable set of propositional symbols, $p \in P$.

$$\varphi := p \mid \neg\varphi \mid \varphi \wedge \varphi \mid \Box\varphi$$

DEFINITION

A **topological model** (or a topo-model) $M = (X, \mathcal{O}, \nu)$ is a topological space $(X, \mathcal{O})$ together with a valuation function $\nu : P \rightarrow \mathcal{P}(X)$.

DEFINITION (SEMANTICS)

Given a topological model $M = (X, \mathcal{O}, \nu)$ and a state $x \in X$:

$M, x \models p$	iff	$x \in \nu(p)$ for each $p \in P$
$M, x \models \neg\varphi$	iff	not $M, x \models \varphi$
$M, x \models \varphi \wedge \psi$	iff	$M, x \models \varphi$ and $M, x \models \psi$
$M, x \models \Box\varphi$	iff	there is $U \in \tau(x \in U$ and for all $y \in U$: $M, y \models \varphi)$

SOUND AND COMPLETE TOPO-AXIOMATIZATIONS

Rules

(MP) if $\vdash \varphi$ and $\vdash \varphi \rightarrow \psi$, then $\vdash \psi$

(N) if $\vdash \varphi$, then $\vdash \Box \varphi$

Axioms

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(4) $\Box\varphi \rightarrow \Box\Box\varphi$

S4 is the topo-logic of all topological spaces (McKinsey & Tarski 1944).

SOUND AND COMPLETE TOPO-AXIOMATIZATIONS

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$S4 = \text{Topo}$

S4 is the topo-logic of all topological spaces (McKinsey & Tarski 1944).

WHAT ABOUT $\mathcal{T}_d$ -SPACES (IDENTIFIABLE IN THE LIMIT)?

$\mathcal{T}_d$ is not topo-definable.

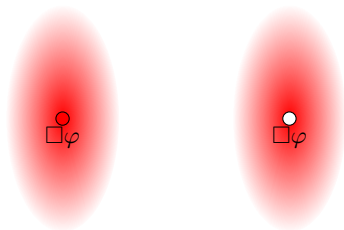
The identifiability-adequate notion of belief is not *topo*-definable.

WHAT ABOUT T_d -SPACES (IDENTIFIABLE IN THE LIMIT)?

T_d is not topo-definable.

The identifiability-adequate notion of belief is not *topo*-definable.

But let us, on a whim, change the way we view $\Box$.



TOPOLOGICAL d -SEMANTICS

DEFINITION (SEMANTICS)

Given a topological model $M = (X, \mathcal{O}, v)$ and a state $x \in X$:

$M, x \models_d p$	iff	$x \in v(p)$
$M, x \models_d \neg\varphi$	iff	not $M, x \models_d \varphi$
$M, x \models_d \varphi \wedge \psi$	iff	$M, x \models_d \varphi$ and $M, x \models_d \psi$
$M, x \models_d \Box\varphi$	iff	$\exists U \in \tau (x \in U \ \& \ \forall y \in U - \{x\} \ M, y \models_d \varphi)$

SOUND AND COMPLETE d -AXIOMATIZATIONS

Rules

(MP) if $\vdash \varphi$ and $\vdash \varphi \rightarrow \psi$, then $\vdash \psi$

(N) if $\vdash \varphi$, then $\vdash \Box \varphi$

Axioms

(K) $\Box(\varphi \rightarrow \psi) \rightarrow (\Box \varphi \rightarrow \Box \psi)$

(T) $\Box \varphi \rightarrow \varphi$

(D) $\Box \varphi \rightarrow \neg \Box \neg \varphi$

(4) $\Box \varphi \rightarrow \Box \Box \varphi$

(5) $\neg \Box \varphi \rightarrow \Box \neg \Box \varphi$

(w) $(\varphi \wedge \Box \varphi) \rightarrow \Box \Box \varphi$

(GL) $\Box(\Box \varphi \rightarrow \varphi) \rightarrow \Box \varphi$

SOUND AND COMPLETE d -AXIOMATIZATIONS

Rules

(MP) if $\vdash \varphi$ and $\vdash \varphi \rightarrow \psi$, then $\vdash \psi$

(N) if $\vdash \varphi$, then $\vdash \Box \varphi$

Axioms

(K) $\Box(\varphi \rightarrow \psi) \rightarrow (\Box \varphi \rightarrow \Box \psi)$

(D) $\Box \varphi \rightarrow \neg \Box \neg \varphi$

(5) $\neg \Box \varphi \rightarrow \Box \neg \Box \varphi$

(w) $(\varphi \wedge \Box \varphi) \rightarrow \Box \Box \varphi$

wKD45=dense

wKD45 is the d-logic of dense spaces.

SOUND AND COMPLETE *d*-AXIOMATIZATIONS

Rules

(MP) if $\vdash \varphi$ and $\vdash \varphi \rightarrow \psi$, then $\vdash \psi$

(N) if $\vdash \varphi$, then $\vdash \Box \varphi$

Axioms

(K) $\Box(\varphi \rightarrow \psi) \rightarrow (\Box \varphi \rightarrow \Box \psi)$

(D) $\Box \varphi \rightarrow \neg \Box \neg \varphi$

(4) $\Box \varphi \rightarrow \Box \Box \varphi$

(5) $\neg \Box \varphi \rightarrow \Box \neg \Box \varphi$

KD45=DSO

KD45 is the d-logic of DSO-spaces.

SOUND AND COMPLETE d -AXIOMATIZATIONS

Rules

(MP) if $\vdash \varphi$ and $\vdash \varphi \rightarrow \psi$, then $\vdash \psi$

(N) if $\vdash \varphi$, then $\vdash \Box \varphi$

Axioms

(K) $\Box(\varphi \rightarrow \psi) \rightarrow (\Box \varphi \rightarrow \Box \psi)$

GL=scattered

(GL) $\Box(\Box \varphi \rightarrow \varphi) \rightarrow \Box \varphi$

GL is the d -logic of scattered spaces.

SOUND AND COMPLETE d -AXIOMATIZATIONS

Rules

(MP) if $\vdash \varphi$ and $\vdash \varphi \rightarrow \psi$, then $\vdash \psi$

(N) if $\vdash \varphi$, then $\vdash \Box \varphi$

Axioms

(K) $\Box(\varphi \rightarrow \psi) \rightarrow (\Box \varphi \rightarrow \Box \psi)$

wK4=Topo

(w) $(\varphi \wedge \Box \varphi) \rightarrow \Box \Box \varphi$

wK4 is the d -logic of all topological spaces.

SOUND AND COMPLETE d -AXIOMATIZATIONS

Rules

(MP) if $\vdash \varphi$ and $\vdash \varphi \rightarrow \psi$, then $\vdash \psi$

(N) if $\vdash \varphi$, then $\vdash \Box \varphi$

Axioms

(K) $\Box(\varphi \rightarrow \psi) \rightarrow (\Box \varphi \rightarrow \Box \psi)$

$K4 = T_d$

(4) $\Box \varphi \rightarrow \Box \Box \varphi$

Finally, K4 is the d-logic of all T_d -spaces!

SOUND AND COMPLETE d -AXIOMATIZATIONS

Rules

(MP) if $\vdash \varphi$ and $\vdash \varphi \rightarrow \psi$, then $\vdash \psi$

(N) if $\vdash \varphi$, then $\vdash \Box \varphi$

Axioms

(K) $\Box(\varphi \rightarrow \psi) \rightarrow (\Box \varphi \rightarrow \Box \psi)$

And so what...?

(4) $\Box \varphi \rightarrow \Box \Box \varphi$

Finally, K4 is the d -logic of all T_d -spaces!

Get dynamic!



Baltag, A., Gierasimczuk, N., Özgün, A., Vargas Sandoval, A.L., and Smets S., A dynamic logic for learning theory. J. Log. Algebr. Meth. Program. 2019.

STARTING POINT: SUBSET SPACE LOGIC

DEFINITION (SYNTAX)

Let P be a countable set of propositional symbols and $p \in P$.

$$\varphi := p \mid \neg\varphi \mid \varphi \wedge \varphi \mid K\varphi \mid \Box\varphi$$

DEFINITION

An **intersection model** $M = (X, \mathcal{O}, \nu)$ is an intersection space $(X, \mathcal{O})$ together with a valuation function $\nu : P \rightarrow \mathcal{P}(X)$.

DEFINITION (SEMANTICS)

Given an intersection model $M = (X, \mathcal{O}, \nu)$, $U \in \mathcal{O}$, and $x \in U$:

$M, x, U \models p$	iff	$x \in \nu(p)$
$M, x, U \models \neg\varphi$	iff	$M, x, U \not\models \varphi$
$M, x, U \models \varphi \wedge \psi$	iff	$M, x, U \models \varphi$ and $M, x, U \models \psi$
$M, x, U \models K\varphi$	iff	$\forall y \in U \ M, y, U \models \varphi$
$M, x, U \models \Box\varphi$	iff	$\forall O \in \mathcal{O} \text{ if } x \in O \subseteq U \text{ then } M, x, O \models \varphi$



A DYNAMIC LOGIC FOR LEARNING THEORY (DLLT)

DEFINITION (SYNTAX)

Let p and o be drawn from countable sets of propositional and observational symbols, P and O respectively.

$$\varphi := p \mid o \mid L(\vec{o}) \mid \neg\varphi \mid \varphi \wedge \varphi \mid K\varphi \mid \Box\varphi \mid [o]\varphi$$

(DLLT): LEARNING MODELS

DEFINITION

A *learning model* $\mathcal{M} = (X, \mathcal{O}, \mathbb{L}, \nu)$ consists of:

- ▶ an intersection space $(X, \mathcal{O})$, as before.
- ▶ a learner $\mathbb{L} : \mathcal{O} \rightarrow \mathcal{P}(X)$, s.t.:
 1. $\mathbb{L}(O) \subseteq O$, and
 2. if $O \neq \emptyset$ then $\mathbb{L}(O) \neq \emptyset$.(Additionally: $\mathbb{L}(\vec{O}) := \mathbb{L}(\bigcap \vec{O})$, where $\bigcap \vec{O} := O_1 \cap \dots \cap O_n$).
- ▶ a valuation map $\nu : P \cup \mathcal{O} \rightarrow \mathcal{P}(X)$

(DLLT): SEMANTICS

DEFINITION (SEMANTICS)

Given a learning model $\mathcal{M} = (X, \mathcal{O}, \mathbb{L}, v)$, $U \in \mathcal{O}$, and $x \in U$:

$M, x, U \models p$	iff	$x \in v(p)$
$M, x, U \models o$	iff	$x \in v(o)$
$M, x, U \models L(o_1, \dots, o_n)$	iff	$x \in \mathbb{L}(U, v(o_1), \dots, v(o_n))$
$M, x, U \models \neg\varphi$	iff	$M, x, U \not\models \varphi$
$M, x, U \models \varphi \wedge \psi$	iff	$M, x, U \models \varphi$ and $M, x, U \models \psi$
$M, x, U \models K\varphi$	iff	$\forall y \in U \ M, y, U \models \varphi$
$M, x, U \models \Box\varphi$	iff	$\forall O \in \mathcal{O}$ if $x \in O \subseteq U$ then $M, x, O \models \varphi$
$M, x, U \models [o]\varphi$	iff	$x \in v(o)$ implies $M, x, U \cap v(o) \models \varphi$

ABBREVIATIONS

- ▶ $\bigwedge \vec{o} := o_1 \wedge \dots \wedge o_n \ (\bigwedge \lambda := \top)$
- ▶ $\vec{o} \Leftrightarrow \vec{u} := K((\bigwedge \vec{o}) \leftrightarrow (\bigwedge \vec{u}))$
- ▶ $[\vec{o}]\varphi := [o_1] \dots [o_n]\varphi \ ([\lambda]\varphi := \varphi)$; similarly for $\langle \vec{o} \rangle$
- ▶ $B^{\vec{o}}\varphi := K(L(\vec{o}) \rightarrow \varphi)$
- ▶ $B\varphi := B^\lambda\varphi$

(DLLT): AXIOMATIZATION

BASIC AXIOMS AND RULES

Basic axioms:

(P)	all instantiations of propositional tautologies
(K _K)	$K(\varphi \rightarrow \psi) \rightarrow (K\varphi \rightarrow K\psi)$
(T _K)	$K\varphi \rightarrow \varphi$
(4 _K)	$K\varphi \rightarrow KK\varphi$
(5 _K)	$\neg K\varphi \rightarrow K\neg K\varphi$
(K _[o])	$[o](\psi \rightarrow \chi) \rightarrow ([o]\psi \rightarrow [o]\chi)$

Basic rules:

(MP)	From φ and $\varphi \rightarrow \psi$, infer ψ
(Nec _K)	From φ , infer $K\varphi$
(Nec _[o])	From φ , infer $[o]\varphi$

(DLLT): AXIOMATIZATION

LEARNING AXIOMS

Learning axioms:

(CC)

$$(\bigwedge \vec{o}) \rightarrow \langle K \rangle L(\vec{o})$$

Consistency

(EC)

$$(\vec{o} \Leftrightarrow \vec{u}) \rightarrow (L(\vec{o}) \Leftrightarrow L(\vec{u}))$$

Extensionality

(SP)

$$L(\vec{o}) \rightarrow \bigwedge \vec{o}$$

Success Postulate

(DLLT): AXIOMATIZATION

REDUCTION AXIOMS

Reduction axioms:

(R_p)	$[o]p \leftrightarrow (o \rightarrow p)$
(R_u)	$[o]u \leftrightarrow (o \rightarrow u)$
(R_L)	$[o]L(\vec{u}) \leftrightarrow (o \rightarrow L(o, \vec{u}))$
$(R_{\neg})$	$[o]\neg\psi \leftrightarrow (o \rightarrow \neg[o]\psi)$
(R_K)	$[o]K\psi \leftrightarrow (o \rightarrow K[o]\psi)$
$(R_{\Box})$	$[o]\Box\psi \leftrightarrow \Box[o]\psi$

Effort axiom and rule:

$(\Box\text{-Ax})$	$\Box\varphi \rightarrow [\vec{o}]\varphi, \text{ for } \vec{o} \in O^*$
$(\Box\text{-Rule})$	From $\psi \rightarrow [o]\varphi$, infer $\psi \rightarrow \Box\varphi$, where $o \notin O_\psi \cup O_\varphi$

COMPLETENESS

THEOREM

DLLT is sound and complete with respect to the class of learning models.

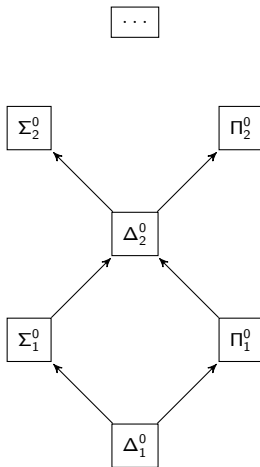
EXPRESSIVITY (OF LEARNING CONCEPTS)

PROPOSITION

- ▶ $\mathcal{M}, x, U \models \Diamond Kp$ iff p is **learnable with certainty** at x .
- ▶ $\mathcal{M} \models p \rightarrow \Diamond Kp$ iff p is **verifiable with certainty**.
- ▶ $\mathcal{M} \models \neg p \rightarrow \Diamond K\neg p$ iff p is **falsifiable with certainty**.
- ▶ $\mathcal{M}, x, U \models \Box Bp$ iff $\mathbb{L}$ has **undefeated belief** in p at x .
- ▶ $\mathcal{M}, x, U \models p \wedge \Box Bp$ iff $\mathbb{L}$ has **inductive knowledge** of p at x .
- ▶ $\mathcal{M}, x, U \models p \wedge \Diamond \Box Bp$ iff p is **inductively learnable** by $\mathbb{L}$ at x .
- ▶ $\mathcal{M} \models p \rightarrow \Diamond \Box Bp$ iff p is **verifiable in the limit** by $\mathbb{L}$.
- ▶ $\mathcal{M} \models \neg p \rightarrow \Diamond \Box B\neg p$ iff p is **falsifiable in the limit** by $\mathbb{L}$.

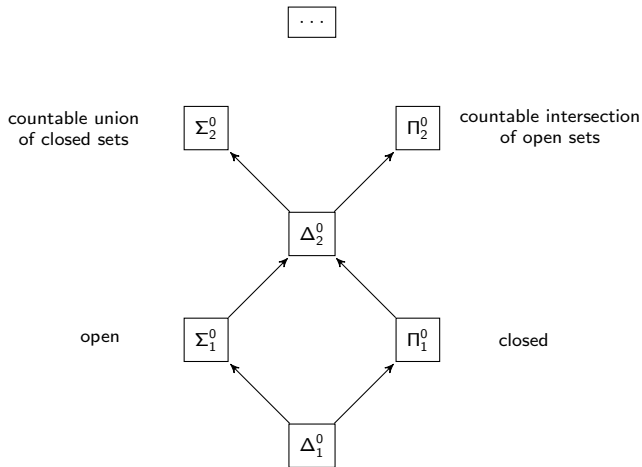
DESCRIPTIVE SET THEORY

BOREL HIERARCHY



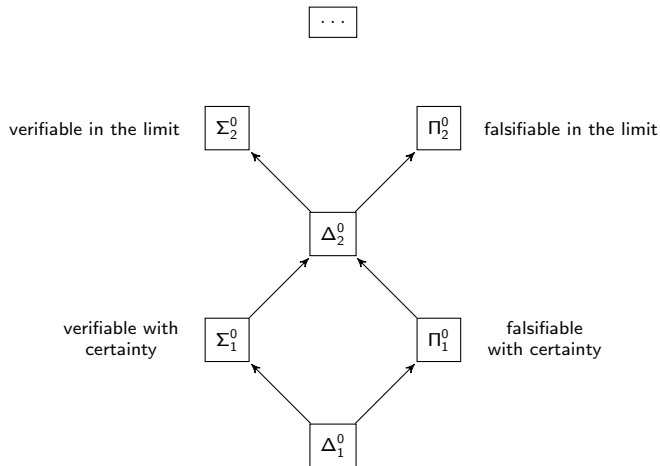
DESCRIPTIVE SET THEORY

BOREL HIERARCHY



DESCRIPTIVE SET THEORY

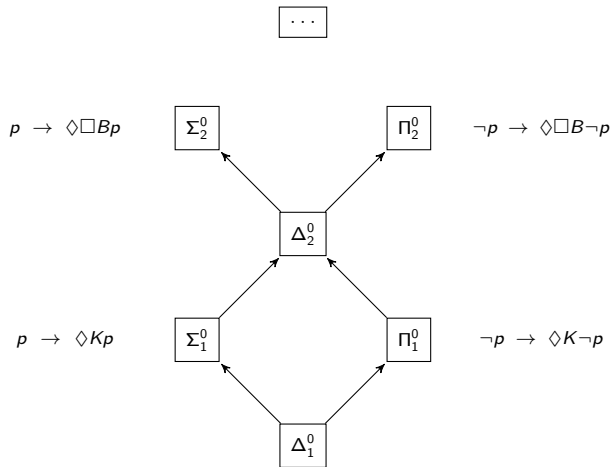
BOREL HIERARCHY



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DESCRIPTIVE SET THEORY

DLIT-DEFINABILITY



CONCLUSIONS

- ▶ Dynamic Modal Logic treatment of learnability.
- ▶ **Topological perspective on knowledge is a bridge between modal logic, learning theory, and computability.**

THE END

Thank you!