Many-sorted logic

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EASLLC, July 2014
Introduction
- several “sorts” of variables
- vector spaces: scalars and vectors
- geometry: points and lines
- second order logic: individuals and subsets.
- elements and sequences of elements
Many-sorted logic

- Can be translated into one-sorted logic
- Many-sorted version of a theorem is sometimes more powerful than the single sorted version.
- Craig Interpolation Theorem.
- Applications of the many-sorted interpolation theorem.
- Applications to "symbiosis" between set theory and model theory.
Single sorted logic
Usual “textbook” predicate logic is single sorted.
Models (i.e. structures) have only one domain (i.e. universe).
All variables range over the same domain.
Definition

An ordinary single sorted first order structure (model)

\[ \mathcal{M} = (M, R_1, \ldots, R_m, f_1, \ldots, f_k, c_1, \ldots, c_l) \]

consists of

1. Universe (domain) \( M \neq \emptyset \)
2. Relations \( R_1, \ldots, R_m \) between elements of the universe
3. Functions \( f_1, \ldots, f_k \) between elements of the universe
4. Distinguished constants \( c_1, \ldots, c_l \) in the universe.

The integers \( m, k, l \) maybe be 0.
where ...

\[ R_i \subseteq M \times \ldots \times M \]  \text{relation}

\[ f_i : M \times \ldots \times M \rightarrow M \]  \text{function}

\[ c_i \in M \]  \text{constant}
Example

- $\langle \{0, 1, 2\}, < \rangle$
- $\langle \mathbb{N}, < \rangle$
- $\langle \mathbb{N}, <, 0 \rangle$
- $\langle \mathbb{N}, s, 0 \rangle$, where $s(n) = n + 1$
- $\langle \{0, 1, 2\}, \{(0, 1), (0, 2), (1, 0), (1, 2), (2, 0), (2, 1)\} \rangle$
Syntax in single sorted logic

<table>
<thead>
<tr>
<th>Symbols</th>
<th>Description</th>
<th>Arity</th>
</tr>
</thead>
<tbody>
<tr>
<td>$R$</td>
<td>relation symbols</td>
<td>$\alpha(R) \in \mathbb{N}$</td>
</tr>
<tr>
<td>$f$</td>
<td>function symbols</td>
<td>$\alpha(f) \in \mathbb{N}$</td>
</tr>
<tr>
<td>$c$</td>
<td>constant symbols</td>
<td>$\alpha(c) = 0$</td>
</tr>
</tbody>
</table>

Sometimes we write just $R$, $f$ and $c$, forgetting the underline.
## Syntax in single sorted logic

<table>
<thead>
<tr>
<th>Syntax</th>
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</tr>
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<tbody>
<tr>
<td>$t = t'$</td>
<td>equational atomic formula</td>
</tr>
<tr>
<td>$Rt_1...t_n$</td>
<td>relational atomic formula</td>
</tr>
<tr>
<td>$\neg \phi$</td>
<td>negation</td>
</tr>
<tr>
<td>$\land \phi$</td>
<td>conjunction (possibly infinite)</td>
</tr>
<tr>
<td>$\lor \phi$</td>
<td>disjunction (possibly infinite)</td>
</tr>
<tr>
<td>$\exists x_i \phi$</td>
<td>existentially quantified formula</td>
</tr>
<tr>
<td>$\forall x_i \phi$</td>
<td>universally quantified formula</td>
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<td>negation</td>
</tr>
<tr>
<td>$\phi_0 \land \phi_1$</td>
<td>conjunction (binary)</td>
</tr>
<tr>
<td>$\phi_0 \lor \phi_1$</td>
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In second order logic add

\[ \exists X^m_i \phi \]

where \( X^m_i \) is an \( m \)-ary relation variable

\[ \forall X^m_i \phi \]

where \( X^m_i \) is an \( m \)-ary relation variable

In **monadic** second order logic \( m = 1 \).

New atomic formulas \( X^m_it_1 \ldots t_m \).
If
\[ \mathcal{M} = (M, R_1, \ldots, R_m, f_1, \ldots, f_k, c_1, \ldots, c_l) \]
then
\[ L = \{ R_1, \ldots, R_m, f_1, \ldots, f_k, c_1, \ldots, c_l \} \]
is called the vocabulary of $\mathcal{M}$, $\text{voc}(\mathcal{M})$. We also write
\[ \mathcal{M} = (M, R_1^\mathcal{M}, \ldots, R_m^\mathcal{M}, f_1^\mathcal{M}, \ldots, f_k^\mathcal{M}, c_1^\mathcal{M}, \ldots, c_l^\mathcal{M}) \]
Assignments $v$ into $\mathcal{M}$ are functions on variables such that $v(x_i) \in M$ for each $x_i$.

Modified assignment $v(a/x_i)$ is defined as follows:

$$v(a/x_i)(x_j) = v(x_j), \text{ if } j \neq i, \text{ and } v(a/x_i)(x_i) = a.$$ 

The value of a term under the assignment $v$:

$$x_i^\mathcal{M} \langle v \rangle = v(x_i), \ C^\mathcal{M} \langle v \rangle = C^\mathcal{M}.$$ 

The value of a term under the assignment $v$:

$$(f^t_1 \ldots t_n)^\mathcal{M} \langle v \rangle = f^\mathcal{M}(t_1^\mathcal{M} \langle v \rangle, \ldots, t_n^\mathcal{M} \langle v \rangle)$$
Semantics in single sorted logic

**Definition (Truth Definition)**

\[ M \models_v t = t' \iff t^M\langle v \rangle = t^M\langle v \rangle. \]

**Example**

- \((\mathbb{N}, <) \models_v x_1 = x_2 \iff v(x_1) = v(x_2)\)
- \((\mathbb{N}, <, 0) \models_v x_1 = 0 \iff v(x_1) = 0\)
- \((\mathbb{N}, s) \models_v ssx_0 = x_1 \iff v(x_0) + 2 = v(x_1), \) where \(s(n) = n + 1\) for all \(n \in \mathbb{N}\).
Definition (Truth Definition)

\[ M \models \mathrel{\forall} R t_1 \ldots t_n \iff (t_1^M(v), \ldots, t_n^M(v)) \in R^M. \]

Example

- \((\mathbb{N}, <) \models \forall x_1 < x_2 \iff v(x_1) < v(x_2)\)
- \((\mathbb{N}, <, 0) \models \forall 0 < x_1 \iff 0 < v(x_1)\)
- \((\mathbb{N}, R) \models \forall R x_0 x_1 \iff v(x_0)^2 = v(x_1), \text{ where } R = \{(n, m) : n^2 = m\} \text{ for all } n, m \in \mathbb{N}.\)
Definition (Truth Definition)

\[ M \models^v \neg \phi \iff M \not\models^v \phi. \]
\[ M \models^v \bigwedge_i \phi_i \iff M \models^v \phi_i \text{ for all } i \in I. \]
\[ M \models^v \bigvee_i \phi_i \iff M \models^v \phi_i \text{ for some } i \in I. \]

Example

1. \((\mathbb{N}, <) \models^v \neg x_1 = x_2 \iff v(x_1) \neq v(x_2)\)
2. \((\mathbb{N}, <, 0) \models^v 0 = x_1 \lor 0 = x_2 \iff v(x_1) = 0 \text{ or } v(x_2) = 0\)
Semantics in single sorted logic

Definition (Truth Definition)

First order logic:

\[ \mathcal{M} \models v \exists x_i \phi \iff \mathcal{M} \models v(a/x_i) \phi \text{ for some } a \in M. \]

\[ \mathcal{M} \models v \forall x_i \phi \iff \mathcal{M} \models v(a/x_i) \phi \text{ for all } a \in M. \]

Example

\((\mathbb{N}, <) \models \exists x_0 \forall x_1 (x_0 < x_1 \lor x_0 = x_1)\)
Definition (Truth Definition)

Second order logic:

\[ \mathcal{M} \models v \exists X_i^m \phi \iff \mathcal{M} \models v(A/X_i^m) \phi \text{ for some } A \subseteq M^m. \]

\[ \mathcal{M} \models v \forall X_i^m \phi \iff \mathcal{M} \models v(a/X_i^m) \phi \text{ for all } A \subseteq M^m. \]

Example

\[ (\mathbb{N}, <) \models \forall X_0^1 (\forall x_0 \neg X_0^1 x_0 \lor \exists x_0 X_0^1 x_0) \]
Many sorted logic: its structures
A many-sorted structure (model) 

\[ \mathcal{M} = (\{M_1, \ldots, M_n\}, R_1, \ldots, R_m, f_1, \ldots, f_k, c_1, \ldots, c_l) \]

consists of

1. Universes ("sorts") \( M_1, \ldots, M_n \), each \( \neq \emptyset \)
2. Relations \( R_1, \ldots, R_m \) between elements of the universes
3. Functions \( f_1, \ldots, f_k \) between elements of the universes
4. Distinguished constants \( c_1, \ldots, c_l \) in the universes.

We call \( \mathcal{M} \) \( n \)-sorted. \( \mathcal{M} \) is "strict" if the \( M_i \) are disjoint, otherwise "lax".

We allow \( n = 0 \), i.e. "no sorts" as a special case.
where ...

\[ R_i \subseteq M_{i_1} \times \ldots \times M_{i_s} \quad \mathcal{s}(R_i) = \langle i_1, \ldots, i_s \rangle \in \{1, \ldots, n\}^s \]

\[ f_i : M_{i_1} \times \ldots \times M_{i_s} \to M_r \quad \mathcal{s}(f_i) = \langle i_1, \ldots, i_s, r \rangle \in \{1, \ldots, n\}^{s+1} \]

\[ c_i \in M_j \quad \mathcal{s}(c_i) = j \in \{1, \ldots, n\} \]

Restriction: we do not allow relations between elements of sort \( s \) or sort \( s' \) and elements of sort \( s'' \).
A lax many-sorted structure
A strict many-sorted structure
Example

An ordinary single-sorted first order structure

\((M, R_1, \ldots, R_m, f_1, \ldots, f_k, c_1, \ldots, c_l)\)

resurrects in this notation as a 1-sorted structure

\((\{M\}, R_1, \ldots, R_m, f_1, \ldots, f_k, c_1, \ldots, c_l)\)
A Chu space is a strict 2-sorted structure \((\{A, X\}, R)\), (denoted \((A, X, R)\)), where \(R \subseteq A \times X\). For example:

<table>
<thead>
<tr>
<th>(A)</th>
<th>(X)</th>
<th>(R_{ax})</th>
</tr>
</thead>
<tbody>
<tr>
<td>elements</td>
<td>sets</td>
<td>(a \in x)</td>
</tr>
<tr>
<td>models</td>
<td>sentences</td>
<td>(a \models x)</td>
</tr>
<tr>
<td>sentences</td>
<td>models</td>
<td>(x \models a) (&quot;dual&quot; of the previous).</td>
</tr>
</tbody>
</table>

Jouko Väänänen (Helsinki and Amsterdam)
Example

A vector space is a strict 2-sorted structure

\[ V = (\{ F, V \}, +_F, \cdot_F, 0_F, 1_F, +_V, \cdot_V, 0_V), \]

where

1. \((F, +_F, \cdot_F, 0_F, 1_F)\) is a field
2. \((V, +_V, 0_V)\) is an Abelian group
3. \(\cdot_V : F \times V \to V\) satisfies
   - \(a \cdot_V (u +_V w) = a \cdot_V u +_V a \cdot_V w\)
   - \((a \cdot_F b) \cdot_V u = a \cdot_V (b \cdot_V u)\)
   - \((a +_F b) \cdot_V u = a \cdot_V u +_V b \cdot_V u\)
   - \(1_F \cdot_V u = u\)
A vector space is a strict 2-sorted structure

\[ \mathcal{V} = (F, V, +, \cdot, 0), \]

where

1. \( F \) is a field
2. \((V, +, 0)\) is an Abelian group
3. The function \((a, v) \mapsto av(= a \cdot v) : F \times V \to V\) satisfies
   - \( a(u + w) = au + aw \)
   - \((ab) \cdot u = a(bu) \)
   - \((a + b)u = au + bu \)
   - \(1u = u \)
Model with a pairing function

Example

A first order structure with a pairing function is a strict 2-sorted structure

\[ M = (\{N, M\}, \pi, ...), \]

where \( \pi \) is a pairing function, i.e. a bijection \( M \times M \rightarrow N \). We can also let \( N = M^2 \) and \( \pi(x, y) = (x, y) \), the ordered pair of \( x \) and \( y \).

Note: If \( M \) is finite, \( \pi \) cannot be a function into \( M \), so a new sort is needed.
Example

A first order structure with finite sequences is a strict 3-sorted structure

\[ M = (\bigcup n M^n, \mathbb{N}, M, \text{len}, 0, s, \sigma \ldots), \]

where \( \text{len}(a) = n \) if \( a \in M^n \), \( s \) is the successor function on \( \mathbb{N} \), and

\[ \sigma(\langle a_1, \ldots, a_n \rangle, i) = a_i. \]
Second order domain

Example

A first order structure with **second order domain** is a strict 2-sorted structure

\[ \mathcal{M} = (\{\mathcal{P}(M), M\}, \in, ...), \]

where

1. \(\in\) is the membership relation \(a \in B\) between \(a \in M\) and \(B \subseteq M\)
2. \((M, ...\) is a first order structure.
Higher order domains

Example

A first order structure with **second and third order domains** is a strict 3-sorted structure

\[ \mathcal{M} = (\{ \mathcal{P}(\mathcal{P}(M)), \mathcal{P}(M), M \}, \in_1, \in_2, \ldots), \]

where

1. \( \in_1 \) is the membership relation \( a \in_1 B \) between \( a \in M \) and \( B \subseteq M \)
2. \( \in_2 \) is the membership relation \( B \in_2 C \) between \( B \subseteq M \) and \( C \subseteq \mathcal{P}(M) \)
3. \( (M, \ldots) \) is a first order structure.
The database

<p>| | | | | |</p>
<table>
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<tr>
<th></th>
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</thead>
<tbody>
<tr>
<td>John</td>
<td>male</td>
<td>D</td>
<td>2500</td>
<td>Mary</td>
</tr>
<tr>
<td>Mary</td>
<td>female</td>
<td>E</td>
<td>2600</td>
<td>Paul</td>
</tr>
<tr>
<td>Laura</td>
<td>female</td>
<td>B</td>
<td>2500</td>
<td>Mary</td>
</tr>
</tbody>
</table>

is a 4-sorted structure with the sorts

1: Name-sort
2: Gender-sort
3: Department-sort
4: Salary-sort
The database is

\[(\{M_1, M_2, M_3, M_4\}, R)\]

where

\[M_1 = \{John, Mary, Laura, Paul\}\]
\[M_2 = \{female, male\}\]
\[M_3 = \{B, D, E\}\]
\[M_4 = \{2500, 2600\}\]

and

\[R \subseteq M_1 \times M_2 \times M_3 \times M_4 \times M_1\]

is the relation indicated by the database.
Example

The two table (relation) relational database

\[
R_1: \begin{array}{llll}
\text{John} & \text{male} & D & \text{Mary} \\
\text{Mary} & \text{female} & E & \text{Paul} \\
\text{Laura} & \text{female} & B & \text{Mary}
\end{array}
\]

\[
R_2: \begin{array}{ll}
\text{A} & 1000 \\
\text{B} & 1300 \\
\text{C} & 1500 \\
\text{D} & 2500 \\
\text{E} & 2600
\end{array}
\]

is another example of a 4-sorted structure, this time with two relations.
More about many sorted structures
Definition

A many-sorted structure

\[ \mathcal{M} = (\{M_1, \ldots, M_n\}, R_1, \ldots, R_m, f_1, \ldots, f_k, c_1, \ldots, c_l) \]

is isomorphic to

\[ \mathcal{M}' = (\{M'_1, \ldots, M'_n\}, R'_1, \ldots, R'_m, f'_1, \ldots, f'_k, c'_1, \ldots, c'_l) \]

if there is a bijection \( \pi : \bigcup_s M_s \rightarrow \bigcup_s M'_s \) such that

1. \( \pi \upharpoonright M_i : M_i \rightarrow M'_i \) is bijection for \( i = 1, \ldots, n \)
2. \( R_i(a_1, \ldots, a_s) \iff R'_i(\pi(a_1), \ldots, \pi(a_s)) \)
3. \( \pi f_i(a_1, \ldots, a_s) = f'_i(\pi(a_1), \ldots, \pi(a_s)) \)
4. \( \pi c_i = c'_i \)
Strict

\[ M_1 \xrightarrow{\pi} M'_1 \]

\[ M_2 \xrightarrow{\pi} M'_2 \]
**Definition**

A many-sorted structure

\[ \mathcal{M} = (\{M_1, \ldots, M_n\}, R_1, \ldots, R_m, f_1, \ldots, f_k, c_1, \ldots, c_l) \]

is an extension of a many-sorted structure

\[ \mathcal{M}' = (\{M'_1, \ldots, M'_n\}, R'_1, \ldots, R'_m, f'_1, \ldots, f'_k, c'_1, \ldots, c'_l) \]

if

1. \( M'_i \subseteq M_i \) for \( i = 1, \ldots, n \)
2. \( R'_i(a_1, \ldots, a_s) \iff R_i(a_1, \ldots, a_s) \) for \( a_1, \ldots, a_s \in \bigcup_{i=1}^n M'_i \)
3. \( f'_i(a_1, \ldots, a_s) = f_i(a_1, \ldots, a_s) \) for \( (a_1, \ldots, a_s) \in \text{dom}(f') \).
4. \( c'_i = c_i \)

Then \( \mathcal{M}' \) is a substructure of \( \mathcal{M} \).
A Chu transform between Chu spaces \((A, X, R)\) and \((A', X', R')\) is a pair \(h, g\) of mapping such that

1. \(h : A \to A'\)
2. \(g : X' \to X\)
3. \(\forall a \in A \forall x' \in X' (R'(h(a), x') \leftrightarrow R(a, g(x'))))\)
Chu transforms

Example

Transforming (monadic) second order logic into first order logic:

- $A$: sentences of single sorted monadic second order logic
- $X$: single sorted structures $(M, ...)$
- $R$: truth
- $A'$: sentences of 2-sorted first order logic
- $X'$: second order domain $(\mathcal{P}(M), M, \in, ...)$
- $R'$: truth
- $h$: translation of $A$ into $A'$
- $g$: the first order part

where

$$h(\exists x_i \phi) = \exists x^0_i h(\phi), \ h(\exists X^1_i \phi) = \exists x^1_i h(\phi)$$
$$g((\mathcal{P}(M), M, \in, ...)) = (M, ...)$$
Chu transforms

Example

- **Continuous maps in topology:**
  - $A$ topological space
  - $X$ its topology
  - $R$ elementhood
  - $A'$ topological space
  - $X'$ its topology
  - $R'$ elementhood
  - $h$ continuous mapping $A \to A'$
  - $g \enspace g(U) = \{ a \in A : f(a) \in U \}$ i.e. preimage

- Now $a \in g(U) \iff h(a) \in U$. 

Chu transforms

Example

- Important special case: $A \subseteq A', X' \subseteq X$, $h = g = id$.

For example, first order arithmetic vs. second order arithmetic.
A reduct of a many-sorted structure

\[ M = (\{M_1, \ldots, M_n\}, R_1, \ldots, R_m, f_1, \ldots, f_k, c_1, \ldots, c_l) \]

is obtained by leaving out some sorts, relations, functions, and constants, as in

\[ M' = (\{M_1, \ldots, M_{n-1}\}, R_1, \ldots, R_{m'}, f_1, \ldots, f_{k'}, c_1, \ldots, c_{l'}) \].

Relations, functions, and constants which are not meaningful are at the same time dropped. Respectively then, \( M \) is an expansion of \( M' \). A reduct may also have the same sorts but fewer relations, etc.
Example

Reducts of a vector space \((F, V, +, \cdot, 0)\):

- The scalar field: \(F = (F, +_F, \cdot_F, 0_F, 1_F)\)
- The vector addition group: \((V, +, 0)\)
Example

Reducts of a second order domain \((\mathcal{P}(M), M, \in, \ldots)\):

- The first order part: \((M, \ldots)\)
- The second order part: \((\mathcal{P}(M), M, \in)\)
Example

A reduct of the database

<p>| | | | |</p>
<table>
<thead>
<tr>
<th></th>
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</thead>
<tbody>
<tr>
<td>John</td>
<td>male</td>
<td>IV</td>
<td>2500</td>
</tr>
<tr>
<td>Mary</td>
<td>female</td>
<td>V</td>
<td>2600</td>
</tr>
<tr>
<td>Laura</td>
<td>female</td>
<td>II</td>
<td>2500</td>
</tr>
<tr>
<td>Paul</td>
<td>Mary</td>
<td></td>
<td></td>
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is not obtained by forgetting (e.g.) the number-sort:

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<td>female</td>
<td>II</td>
<td>Mary</td>
</tr>
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This is a completely new relation, a so-called projection (not reduct) of the original. In a reduct the entire relation $R$ would drop out.
Example

A reduct of the two table (relation) relational database

<table>
<thead>
<tr>
<th></th>
<th>John</th>
<th>Mary</th>
<th>Paul</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>male</td>
<td>female</td>
<td>D</td>
</tr>
<tr>
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<td>female</td>
<td>E</td>
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Relation $R_1$:

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<tbody>
<tr>
<td>A</td>
<td>1000</td>
<td>1300</td>
</tr>
<tr>
<td>B</td>
<td>1500</td>
<td>2500</td>
</tr>
<tr>
<td>C</td>
<td>2600</td>
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</table>

obtained by forgetting the number-sort is

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</thead>
<tbody>
<tr>
<td>A</td>
<td>male</td>
<td>D</td>
</tr>
<tr>
<td>B</td>
<td>female</td>
<td>E</td>
</tr>
</tbody>
</table>

Relation $R_2$ drops out as it uses the abandoned number-sort.
Definition

A single sorted reduct of a many sorted $L$-structure

$$\mathcal{M} = (\{ M_s : s \in \text{sort}(L) \}, R_1, \ldots, R_m, f_1, \ldots, f_k, c_1, \ldots, c_l)$$

is any reduct

$$\mathcal{M}' = (\{ M_s \}, R_1, \ldots, R_{m'}, f_1, \ldots, f_{k'}, c_1, \ldots, c_{l'})$$

to an $L' = \{ s \}$, where $s \in \text{sort}(L)$. Relations, functions, and constants which are not meaningful are dropped.

Sometimes the single sorted reducts determine the many sorted structure, but usually not (because the single sorted reducts disregard the interaction between sorts.)
Fundamental translation

Definition

The fundamental translation of a relational many-sorted structure

\[ \mathcal{M} = (\{M_1, \ldots, M_n\}, R_1, \ldots, R_m, c_1, \ldots, c_l) \]

is the single sorted expanded structure

\[ \mathcal{M}^* = (M_1 \cup \ldots \cup M_n, M_1, \ldots, M_n, R_1, \ldots, R_m, c_1, \ldots, c_l) \]

where \( M_1, \ldots, M_n \) are new unary predicates (i.e. relations).

From the single sorted \( \mathcal{M}^* \) the original \( \mathcal{M} \) can be readily recovered. We have left function symbols out because there is no satisfactory way to extend functions from individual sorts to the union of all sorts. Anyway, functions can be transformed to relations and then there is no problem.
Example

- $(\{\mathcal{P}(M), M\}, \in, \ldots)^* = (\mathcal{P}(M) \cup M, \mathcal{P}(M), M, \in, \ldots)$
- $\mathcal{M} \cong \mathcal{M}' \iff \mathcal{M}^* \cong \mathcal{M}'^*$

For vector spaces we could define the fundamental translation
$(\{F, V\}, +, 0)^* = (F \cup V, F, V, +, 0)$ by letting the value of the functions $+_F, \cdot_F, +_V$ and $\cdot_V$ to be 0 whenever they are not canonically defined.

Note: $\mathcal{M} \not\cong \mathcal{M}^*$
Syntax and semantics of many sorted logic
Definition

The **vocabulary** $L = \text{voc}(\mathcal{M})$ of a many-sorted structure (model) $\mathcal{M} = (\{M_1, \ldots, M_n\}, R_1^\mathcal{M}, \ldots, R_m^\mathcal{M}, F_1^\mathcal{M}, \ldots, F_k^\mathcal{M}, c_1^\mathcal{M}, \ldots, c_l^\mathcal{M})$ consists of

1. Sort symbols $s_1, \ldots, s_n$
2. Relation symbols $R_1, \ldots, R_m$
3. Function symbols $F_1, \ldots, F_k$
4. Constant symbols $c_1, \ldots, c_l$

We write $L = \{s_1, \ldots, s_n, R_1, \ldots, R_m, F_1, \ldots, F_k, c_1, \ldots, c_l\}$,

$\text{sort}(L) = \{s_1, \ldots, s_n\}$, $\text{rel}(L) = \{R_1, \ldots, R_m\}$, $\text{fun}(L) = \{F_1, \ldots, F_k\}$,

$\text{con}(L) = \{c_1, \ldots, c_l\}$. 
Each vocabulary $L$ has an \textit{arity-function}

$$
\alpha_L : \text{rel}(L) \cup \text{fun}(L) \rightarrow \mathbb{N}
$$

which tells the arity of each predicate and function symbol, and a \textit{sort-function} $s_L$:

$$
s_L(R) \in \text{sort}(L)^{\alpha_L(R)},
\ s_L(f) \in \text{sort}(L)^{\alpha_L(f)} \times \text{sort}(L),
\ s_L(c) \in \text{sort}(L).
$$

We do \textbf{not} have symbols for abstract relations between elements of arbitrary sorts except identity $=$ in lax structures.
Definition

Suppose $L$ is a many sorted vocabulary. The $L$-variables over many-sorted $L$-structures are denoted:

$$x^s, s \in \text{sort}(L)$$

with the intuition (fulfilled in the semantics below) that $x^s$ ranges over the universe $M_s$ of an $L$-structure

$$\mathcal{M} = (\{M_1, \ldots, M_n\}, R^\mathcal{M}_1, \ldots, R^\mathcal{M}_m, F^\mathcal{M}_1, \ldots, F^\mathcal{M}_k, c^\mathcal{M}_1, \ldots, c^\mathcal{M}_l).$$

No variables ranging over “everything”! No variables ranging over elements of two different sort $s$ (but this can be arranged by having a new sort for the “union”).
Definition

Suppose $L$ is a many sorted vocabulary. The $L$-terms are defined as follows:

1. Constants $c$ in $L$ are $L$-terms and $s(c)$ is already defined.
2. $L$-variables $x^s$ are $L$-terms and $s(x^s) = s$.
3. If $t_1, \ldots, t_n$ are $L$-terms with $s_i = s(t_i)$ and $f \in L$ with $s(f) = \langle s_1, \ldots, s_n, s \rangle$, then $ft_1 \ldots t_n$ (or $f(t_1, \ldots, t_n)$) is an $L$-term and $s(ft_1 \ldots t_n) = s$. 
Syntax

- **Strict** many-sorted logic: we allow $t = t'$ only if $s(t) = s(t')$. We consider identity across sorts “meaningless”.
- **Lax** many-sorted logic: we allow $t = t'$ for all terms. We consider identity across sorts “meaningful”.

\[
\begin{align*}
t &= t' \\
R_{t_1 \ldots t_n} \\
\neg \phi \\
\bigwedge_n \phi_n \\
\bigvee_n \phi_n \\
\exists x^s \phi \\
\forall x^s \phi
\end{align*}
\]
Assignments $\nu$ into $\mathcal{M}$ are functions on $\text{voc}(L)$-variables such that $\nu(x^s) \in M_s$ for each $s \in \text{sort}(\mathcal{M})$. Modified assignment $\nu(a/x^s)$ as usual.

**Definition**

\begin{align*}
\mathcal{M} \models_{\nu} \exists x^s \phi & \iff \mathcal{M} \models_{\nu(a/x^s)} \phi \text{ for some } a \in M_s. \\
\mathcal{M} \models_{\nu} \forall x^s \phi & \iff \mathcal{M} \models_{\nu(a/x^s)} \phi \text{ for all } a \in M_s.
\end{align*}
Some propositional axioms of $L_{\omega_1\omega}$

\[
\begin{align*}
\Lambda_n \phi_n & \quad \phi_i \\
\phi_i & \\
\phi_0 & \ldots \phi_i \ldots \\
\Lambda_n \phi_n & \\
\end{align*}
\]

\[
\begin{align*}
\phi_i & \\
\begin{array}{c}
\phi_0 \quad \vdots \quad \phi_i \quad \vdots \\
\end{array} & \\
\begin{array}{c}
[\phi_0] \quad \vdots \quad [\phi_i] \\
\end{array} & \\
\begin{array}{c}
\land_n \phi_n \quad \psi \quad \vdots \quad \psi \quad \vdots \\
\end{array} & \\
\begin{array}{c}
[\phi_0] \quad \vdots \quad [\phi_i] \\
\end{array} & \\
\begin{array}{c}
\land_n \phi_n \quad \psi \quad \vdots \quad \psi \quad \vdots \\
\end{array} & \\
\end{align*}
\]
Quantifier axioms of $L_{\omega_1 \omega}$

<table>
<thead>
<tr>
<th>∀$x^s \phi(x^s)$</th>
<th>$\phi(t)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\phi(t)$</td>
<td>$\exists x^s \phi(x^s)$</td>
</tr>
</tbody>
</table>

(Where $s(t) = s$) (Where $s(t) = s$)

$\phi(x^s)$ $\forall x^s \phi(x^s)$ $\exists x^s \phi(x^s)$ $\psi$

[\phi(x^s)] $\vdots$ $\psi$
The fundamental translation
We define a translation $\phi \mapsto \phi^*$ from many sorted logic into single sorted logic.

We then show: $\mathcal{M} \models \phi \iff \mathcal{M}^* \models \phi^*$
The fundamental translation into single-sorted logic

<table>
<thead>
<tr>
<th>Definition</th>
</tr>
</thead>
<tbody>
<tr>
<td>Suppose $\text{sort}(L) = {s_1, \ldots, s_n}$. Let $L^* = \text{rel}(L) \cup \text{con}(L) \cup {P_1, \ldots, P_n}$, where $P_1, \ldots, P_n$ are new unary predicates.</td>
</tr>
<tr>
<td>1. $(x^{s_i})^<em>$ is $x^i$, $c^</em>$ is $c$</td>
</tr>
<tr>
<td>2. $(t = t')^<em>$ is $t^</em> = t'^<em>$, $(Rt_1 \ldots t_m)^</em>$ is $Rt_1^* \ldots t_m^*$</td>
</tr>
<tr>
<td>3. $(\neg \phi)^<em>$ is $\neg \phi^</em>$, $(\wedge_n \phi_n)^<em>$ is $\wedge_n \phi_n^</em>$, $(\vee_n \phi_n)^<em>$ is $\vee_n \phi_n^</em>$</td>
</tr>
<tr>
<td>4. $(\exists x^{s_i} \phi)^<em>$ is $\exists x^i (P_i(x^i) \wedge \phi^</em>)$</td>
</tr>
<tr>
<td>5. $(\forall x^{s_i} \phi)^<em>$ is $\forall x^i (P_i(x^i) \rightarrow \phi^</em>)$</td>
</tr>
</tbody>
</table>

---

\*We assume $L$ is relational.
The fundamental translation of many sorted logic into single sorted logic

**Definition**

If \( \nu \) is an assignment into \( \mathcal{M} \), let \( \nu^* \) be the following assignment into \( \mathcal{M}^* \):

\[
\nu^*(x^i) = \nu(x^{S_i}).
\]

Note: \( (\nu(a/x^{S_i}))^* = \nu^*(a/x^i) \).

**Proposition**

\( \mathcal{M} \models \nu \phi \iff \mathcal{M}^* \models \nu^* \phi^* \)

**Proof.**

Easy induction on \( \phi \).
\begin{itemize}
  \item \textbf{L*}-formulas need not be of the form $\phi^*$. \\
  In \textit{lax} many sorted logic $\mathcal{M}^*$ always satisfies \\
  \begin{equation*}
    \forall x (\bigvee P_i(x)) \land \bigwedge_i \exists x P_i x.
  \end{equation*}
  \item \textbf{L*}-models of the above are always of the form $\mathcal{M}^*$. \\
  In \textit{strict} many sorted logic $\mathcal{M}^*$ always satisfies \\
  \begin{equation*}
    \forall x (\bigvee P_i(x)) \land \bigwedge_i \exists x P_i x \land \bigwedge_{i \neq j} \neg \exists x (P_i(x) \land P_j(x)).
  \end{equation*}
  \item \textbf{L*}-models of the above are always of the form $\mathcal{M}^*$, where $\mathcal{M}$ is a many sorted structure in the \textit{strict} sense. \\
  The transform $\phi \mapsto \phi^*$ is a Chu transform between the obvious Chu spaces. \\
  \item $\phi \in L_{\omega\omega} \iff \phi^* \in L_{\omega\omega}$, $\phi \in L_{\omega_1\omega} \iff \phi^* \in L_{\omega_1\omega}$
\end{itemize}
Corollary

The following hold both in the lax and in the strict case (even if we allow function symbols—exercise):

1. **Compactness Theorem**: If $T$ is a set of sentences of many sorted first order logic and every finite subset of $T$ has a model, then $T$ has a model.

2. **Löwenheim-Skolem Theorem**: If $T$ is a set of sentences of many sorted first order logic and $T$ has an infinite model, then $T$ has models of all infinite cardinalities $\geq |T|$.

3. **Löwenheim-Skolem Theorem**: If $T$ is a countable of sentences of many sorted $L_{\omega_1\omega}$ and $T$ has an infinite model, then $T$ has a countable model.

4. **Löwenheim-Skolem-Tarski Theorem**: If $\mathcal{M}$ is a many sorted structure with countable vocabulary and $X \subseteq \bigcup_s M_s$ is infinite, then there is $\mathcal{M}' \prec L_{\omega_\omega}$ $\mathcal{M}$ such that $X \subseteq \bigcup_s M'_s$ and $|\bigcup_s M'_s| = |X|$.

5. **Abstract Completeness Theorem**: The set of Gödel numbers of valid many sorted first order sentences is recursively (aka computably) enumerable.
Model construction
Pushing negation inside

\[ \phi \neg \quad = \quad \neg \phi \quad \text{if } \phi \text{ atomic} \]

\[ (\neg \phi) \neg \quad = \quad \phi \]

\[ (\bigwedge_n \phi_n) \neg \quad = \quad \bigvee_n \neg \phi_n \]

\[ (\bigvee_n \phi_n) \neg \quad = \quad \bigwedge_n \neg \phi_n \]

\[ (\forall x^s \phi) \neg \quad = \quad \exists x^s \neg \phi \]

\[ (\exists x^s \phi) \neg \quad = \quad \forall x^s \neg \phi \]
\( \neg \phi \) and \( \phi \neg \) are provably equivalent

**Lemma**

\[ \vdash \neg \phi \leftrightarrow \phi \neg \]

**Proof.**

Exercise.
Model constructions

- For $L_{\omega\omega}$ we have the **Compactness Theorem**, but it is often too rough. For $L_{\omega_1\omega}$ we do not have compactness.
- More flexible: **Consistency Properties**.
- Elaboration of **Beth tableaux**.
- A bit like “forcing”.
- First an auxiliary tool: **Hintikka sets**.
Definition

$L$ countable, $C^s$ a countable set of new constant symbols for $s \in \text{sort}(L)$ and $L' = L \cup C^*$, $C^* = \bigcup_s C^s$. A (lax) **Hintikka set** is any set $H$ of $L'$-sentence of $L_{\omega_1\omega}$ (or $L_{\omega\omega}$), which satisfies:

1. $t = t \in H$ for every constant $L'$-term $t$. (Add $\neg c = c'$ for strict Hintikka set whenever $s(c) \neq s(c')$.)
2. If $\phi(t) \in H$, $\phi(t)$ atomic, and $t = t' \in H$, then $\phi(t') \in H$.
3. If $\neg \phi \in H$, then $\phi \neg \in H$.
4. If $\bigvee_n \phi_n \in H$, then $\phi_n \in H$ for some $n$.
5. If $\bigwedge_n \phi_n \in H$, then $\phi_n \in H$ for all $n$.
6. If $\exists x^s \phi(x^s) \in H$, then $\phi(c) \in H$ for some $c \in C^s$.
7. If $\forall x^s \phi(x^s) \in H$, then $\phi(c) \in H$ for all $c \in C^s$.
8. For every constant $L'$-term $t$ there is $c \in C^s$, $s = s(t)$, such that $t = c \in H$.
9. There is no atomic sentence $\phi$ such that $\phi \in H$ and $\neg \phi \in H$.

The Hintikka set $H$ is a Hintikka set **for** a sentence $\phi$ of $L_{\omega_1\omega}$ (or $L_{\omega\omega}$) if $\phi \in H$. 
1. \( t = t \in H \) for every constant \( L' \)-term \( t \). (Add \( \neg c = c' \) for strict Hintikka set whenever \( s(c) \neq s(c') \).)

2. If \( \phi(t) \in H \), \( \phi(t) \) atomic, and \( t = t' \in H \), then \( \phi(t') \in H \).

3. If \( \neg \phi \in H \), then \( \phi \neg \in H \).

4. If \( \bigvee_n \phi_n \in H \), then \( \phi_n \in H \) for some \( n \).

5. If \( \bigwedge_n \phi_n \in H \), then \( \phi_n \in H \) for all \( n \).

6. If \( \exists x^s \phi(x^s) \in H \), then \( \phi(c) \in H \) for some \( c \in C^s \).

7. If \( \forall x^s \phi(x^s) \in H \), then \( \phi(c) \in H \) for all \( c \in C^s \).

8. For every constant \( L' \)-term \( t \) there is \( c \in C^s \), \( s = s(t) \), such that \( t = c \in H \).

9. There is no atomic \( \phi \) such that \( \phi \in H \) and \( \neg \phi \in H \).
A basic lemma

**Lemma**

If there is a Hintikka set for $\phi \in L_{\omega_1\omega}$ (or $L_{\omega\omega}$), then $\phi$ has a model. If the Hintikka set is strict, then $\phi$ has a strict model.

Note: Conversely, if $\phi$ has a model $\mathcal{M}$, then there is a Hintikka set for $\phi$, built directly from formulas true in $\mathcal{M}$.

Note: A Hintikka set need not be complete. There may be $\phi$ such that neither $\phi \in H$ nor $\neg\phi \in H$. 
Proof.

- Define on $C^* = \bigcup_s C^s$: $c \sim c'$ if $c = c' \in H$.
- $M_s = \{[c] : c \in C^s\}$.
- $c^M = [c]$.
- Let $f^M([c_1], \ldots, [c_n]) = [c]$ for $c \in C^*$ such that $f(c_1, \ldots, c_n) = c \in H$.
- For any constant term $t$ of sort $s$ there is a $c \in C^s$ such that $t = c \in H$. It is easy to see that $t^M = [c]$.
- We let $\mathcal{M} \models R(t_1, \ldots, t_n)$ if and only if $R(t_1, \ldots, t_n) \in H$.
- By induction on $\phi(x_1, \ldots, x_n)$: if $d_1 \ldots, d_n \in C^*$ then:
  1. If $\phi(d_1, \ldots, d_n) \in H$, then $\mathcal{M} \models \phi(d_1, \ldots, d_n)$.
  2. If $\neg \phi(d_1, \ldots, d_n) \in H$, then $\mathcal{M} \not\models \phi(d_1, \ldots, d_n)$.

In particular, $\mathcal{M} \models \phi$ for the $\phi$ we started with, since $\phi \in H$.

Note: In the strict case $s \neq s'$ implies $M_s \cap M_{s'} = \emptyset$. 
How to find useful Hintikka sets?

The tool is: **consistency property**.

A consistency property is a set of (usually) finite sets $S$ which are consistent and the consistency property has information about how to extend $S$ to a Hintikka set, which will then give a model for $S$. 
Definition

Let $L$ be a countable vocabulary, $C^s$ a countable set of new constant symbols for $s \in \text{sort}(L)$ and $L' = L \cup C^*$, $C^* = \bigcup_s C^s$. A \textit{consistency property} is any set $\Delta$ of countable sets $S$ of $L$-formulas of $L_{\omega_1 \omega}$ (or $L_{\omega \omega}$), which satisfies the conditions:

1. If $S \in \Delta$, then $S \cup \{ t = t \} \in \Delta$ for every constant $L'$-term $t$. For \textit{strict Consistency Property} demand $S \cup \{ \neg t = t' \} \in \Delta$ if $s(t) \neq s(t')$.

2. If $\phi(t) \in S \in \Delta$, $\phi(t)$ atomic, and $t = t' \in S$, then $S \cup \{ \phi(t') \} \in \Delta$.

3. If $\neg \phi \in S \in \Delta$, then $S \cup \{ \neg \phi \} \in \Delta$.

4. If $\forall_n \phi_n \in S \in \Delta$, then $S \cup \{ \phi_n \} \in \Delta$ for some $n$.

5. If $\exists_n \phi_n \in S \in \Delta$, then $S \cup \{ \phi_n \} \in \Delta$ for all $n$.

6. If $\exists x^s \phi(x^s) \in S \in \Delta$, then $S \cup \{ \phi(c) \} \in \Delta$ for some $c \in C^s$.

7. If $\forall x^s \phi(x^s) \in S \in \Delta$, then $S \cup \{ \phi(c) \} \in \Delta$ for all $c \in C^s$.

8. For every constant $L'$-term $t$, $s(t) = s$ there is $c \in C^s$ such that $S \cup \{ t = c \} \in \Delta$.

9. There is no atomic formula $\phi$ such that $\phi \in S$ and $\neg \phi \in S$.

The consistency property $\Delta$ is a consistency property \textit{for} a set $T$ of infinitary $L$-sentences if for all $S \in \Delta$ and all $\phi \in T$ we have $S \cup \{ \phi \} \in \Delta$. 
Fragments - to obtain a countability condition.

**Definition**

Let $L$ be a vocabulary. An $L$-**fragment** of $L_{\omega_1\omega}$ is any set $\mathcal{F}$ of formulas of $L_{\omega_1\omega}$ in the vocabulary $L$ such that

1. $\mathcal{F}$ is closed substitutions of terms.
2. $\mathcal{F}$ contains the atomic $L$-formulas.
3. $\neg \varphi \in \mathcal{F}$ if and only if $\varphi \in \mathcal{F}$.
4. $\land \Phi \in \mathcal{F}$ if $\Phi \subseteq \mathcal{F}$ is finite.
5. $\lor \Phi \in \mathcal{F}$ if $\Phi \subseteq \mathcal{F}$, is finite.
6. $\land \Phi \in \mathcal{F}$ if and only if $\lor \Phi \in \mathcal{F}$, and then $\Phi \subseteq \mathcal{F}$.
7. $\forall x^s \varphi \in \mathcal{F}$ if and only if $\varphi \in \mathcal{F}$.
8. $\exists x^s \varphi \in \mathcal{F}$ if and only if $\varphi \in \mathcal{F}$.

Note that a fragment is necessarily closed under subformulas.
Existence of fragments

Lemma

Suppose $\varphi \in L_{\omega_1 \omega}$ with a countable vocabulary $L$. Then there is a countable fragment $\mathcal{F} \subseteq L_{\omega_1 \omega}$ such that $\varphi \in \mathcal{F}$. 
Proof.

Let $\mathcal{F}_0$ consist of atomic $L$-formulas and the formula $\varphi$. Let

$$
\mathcal{F}_{n+1} = \mathcal{F}_n \cup \{\phi(t'_1, \ldots, t'_n) : \phi(t_1, \ldots, t_n) \in \mathcal{F}_n\}
$$

$$
\cup \{\psi : \neg \psi \in \mathcal{F}_n\} \cup \{\neg \psi : \psi \in \mathcal{F}_n\}
$$

$$
\cup \{\psi : \psi \in \Phi, \lor \Phi \in \mathcal{F}_n \text{ or } \land \Phi \in \mathcal{F}_n\}
$$

$$
\cup \{\land \Phi : \Phi \subseteq \mathcal{F}_n \text{ finite}\} \cup \{\lor \Phi : \Phi \subseteq \mathcal{F}_n \text{ finite}\}
$$

$$
\cup \{\psi : \exists y^s \psi \in \mathcal{F}_n \text{ or } \forall y^s \psi \in \mathcal{F}_n\}
$$

$$
\cup \{\exists y^s \psi : \psi \in \mathcal{F}_n\} \cup \{\forall y^s \psi : \psi \in \mathcal{F}_n\}
$$

$$
\cup \{\lor \Phi : \land \Phi \in \mathcal{F}_n\} \cup \{\land \Phi : \lor \Phi \in \mathcal{F}_n\}
$$

and

$$
\mathcal{F} = \bigcup_{n=0}^{\infty} \mathcal{F}_n.
$$
Existence of Hintikka sets

**Lemma**

Let $T$ be a countable set of $L$-sentences of $L_{\omega_1\omega}$ or $L_{\omega\omega}$ and $\mathcal{F}$ a countable fragment such that $T \subseteq \mathcal{F}$. Suppose $\Delta$ is a consistency property for $T$. Then for any $S \in \Delta$ there is a Hintikka set $H$ for $T$ such that $S \subseteq H$. If the consistency property is strict, then so is the Hintikka set.

Jouko Väänänen (Helsinki and Amsterdam)

Many-sorted logic

EASLLC, July 2014
Proof:

Let $Trm$ the set of all constant $L'$-terms.

Let $T = \{ \phi_n : n \in \mathbb{N} \}$

$C^s = \{ c^s_n : n \in \mathbb{N} \}$

$Trm = \{ t_n : n \in \mathbb{N} \}$.

Let $\{ \psi_n : n \in \mathbb{N} \}$ be a list of all elements of $\mathcal{F}$.

We define $H$ as the union of an increasing sequence $S_0, S_1, \ldots$, where $S_0 = S$.

$S_{n+1} = S_n$, but:
1. If $n = 3^i$, then $S_{n+1}$ is $S_n \cup \{\phi_i\} \in \Delta$.

2. If $n = 2 \cdot 3^i$, then $S_{n+1}$ is $S_n \cup \{t_i = t_i\} \in \Delta$. (Additional condition for the strict case.)

3. If $n = 4 \cdot 3^i \cdot 5^j$, $\psi_i = (t = t') \in S_n$, and $\psi_j = \phi(t) \in S_n$ with $\phi(t)$ atomic, then $S_{n+1}$ is $S_n \cup \{\phi(t')\} \in \Delta$.

4. If $n = 8 \cdot 3^i$ and $\psi_i = \neg\psi \in S_n$, then $S_{n+1}$ is $S_n \cup \{\psi\neg\}$.  

5. If $n = 16 \cdot 3^i$ and $\psi_i = \bigvee_m \psi_m \in S_n$, then $S_{n+1}$ is $S_n \cup \{\psi_m\}$ for some $m$, whichever is in $\Delta$.

6. If $n = 32 \cdot 3^i \cdot 5^j$, $j \in \{0, 1\}$ and $\psi_i = \bigwedge_m \psi_m \in S_n$, then $S_{n+1}$ is $S_n \cup \{\psi_j\}(\in \Delta)$.

7. If $n = 64 \cdot 3^i$ and $\psi_i = \exists x^s \phi \in S_n$, then $S_{n+1}$ is $S_n \cup \{\phi(c)\}$ for such $c \in C^s$ that $S_{n+1} \in \Delta$.

8. If $n = 128 \cdot 3^i \cdot 5^j$ and $\psi_i = \forall x^s \phi \in S_n$, then $S_{n+1}$ is $S_n \cup \{\phi(c^s_j)\}(\in \Delta)$.

9. If $n = 256 \cdot 3^i$, then $S_{n+1}$ is $S_n \cup \{t_i = c\}$ for such $c \in C^s(t_i)$ that $S_{n+1} \in \Delta$.

Clearly $\bigcup_n S_n$ is a Hintikka set for $T$. 
Consistency property from “consistency"

**Lemma**

The set $\Delta$ of finite sets $S$ of formulas in $L_{\omega\omega}$ (or $L_{\omega_1\omega}$) such that only finitely many constants from $C^*$ occur in $S$ and $S \not\models \bot$ is a consistency property.

**Proof:**

1. Clearly, if $S \in \Delta$, then $S \cup \{t = t\} \in \Delta$ for every constant $L'$-term $t$.
2. Clearly, if $\phi(t) \in S \in \Delta$, $\phi(t)$ atomic, and $t = t' \in S$, then $S \cup \{\phi(t')\} \in \Delta$.
3. Suppose $\neg\phi \in S \in \Delta$, but $S \cup \{\phi\neg\} \models \bot$. Then $S \models \bot$, contradiction.
Suppose \( \bigvee_n \phi_n \in S \in \Delta \) but \( S \cup \{ \phi_n \} \vdash \bot \) for all \( n \). Then \( S \vdash \bot \), contradiction.

Suppose \( \bigwedge_n \phi_n \in S \in \Delta \) but \( S \cup \{ \phi_n \} \vdash \bot \) for some \( n \). Then \( S \vdash \bot \), contradiction.

Suppose \( \exists x^s \phi(x^s) \in S \in \Delta \) but \( S \cup \{ \phi(c) \} \vdash \bot \) for all \( c \in C^s \). Then \( S \vdash \bot \), because we can choose \( c \) so that it does not occur in \( S \). Contradiction.

Suppose \( \forall x^s \phi(x^s) \in S \in \Delta \) but \( S \cup \{ \phi(c) \} \vdash \bot \) for some \( c \in C^s \). Then \( S \vdash \bot \), contradiction.

Let us consider a constant \( L' \)-term \( t \). There is \( c \in C^s(t) \) such that \( S \cup \{ t = c \} \in \Delta \).

There is no atomic formula \( \phi \) such that \( \phi \in S \) and \( \neg \phi \in S \), because \( \{ \phi, \neg \phi \} \vdash \bot \).
The Completeness Theorem for $L_{\omega\omega}$ or $L_{\omega_1\omega}$

**Theorem**

TFAE for $(\phi \in L_{\omega\omega}$ or) $\phi \in L_{\omega_1\omega}$:

1. $\models \phi$ i.e. $\phi$ is true in all models.
2. $\vdash \phi$ i.e. $\phi$ has a proof.

**Proof.**

If $\phi$ has a proof, then clearly $\models \phi$. If $\phi$ does not have a proof, then $\neg \phi \not\models \bot$. So $\{\neg \phi\} \in \Delta$ for the $\Delta$ in the previous lemma. Hence $\neg \phi$ has a model and $\not\models \phi$. 

[Proof box]
Interpolation
Craig Interpolation Theorem for $L_{\omega_1 \omega}$ (or $L_{\omega \omega}$)

**Theorem**

We assume that $L_1$ and $L_2$ are vocabularies. Suppose $\models \phi \rightarrow \psi$, where $\phi$ is an $L_1$-sentence and $\psi$ is an $L_2$-sentence of $L_{\omega \omega}$. Then there is an $L_1 \cap L_2$-sentence $\theta$ ("interpolant") of $L_{\omega \omega}$ such that

1. $\models \phi \rightarrow \theta$
2. $\models \theta \rightarrow \psi$

We will prove a stronger statement in a moment. Before the statement of the stronger result we look at some consequences of the above.
Proposition

Suppose $\phi$ depends only on $R$ in the sense that

$$M \models \phi \iff N \models \phi$$

whenever $M$ and $N$ have the same domain and the same interpretation of $R$. Then $\models \phi \leftrightarrow \psi$ where no non-logical symbols except $R$ occurs in $\psi$.

Proof.

Let $S_1, \ldots, S_n$ be the non-logical symbols in $\phi$ or $\psi$ in addition to $R$. Let $\phi'$ be the result of replacing each $S_i$ in $\phi$ by a new symbol $S'_i$. Now

$$\models \phi \to \phi'.$$

By the Interpolation Theorem there is $\theta$ containing only $R$ such that $\models \phi \to \theta$ and $\models \theta \to \phi'$. Hence $\models \phi \leftrightarrow \theta$. 

$\square$
Theorem (Beth Definability Theorem)

Suppose the predicate $S$ depends only on $R$ in the sense that

$$\phi(R, S) \land \phi(R, S') \models \forall x (S(x) \leftrightarrow S'(x)).$$

Then there is $\theta(R, x)$ where $S$ does not occur such that

$$\phi(R, S) \models \forall x (S(x) \leftrightarrow \theta(R, x)).$$

Proof.

By assumption

$$\models (\phi(R, S) \land S(c)) \rightarrow (\phi(R, S') \rightarrow S'(c)).$$

Let $\theta(R, c)$ be such that

$$\models (\phi(R, S) \land S(c)) \rightarrow \theta(R, c) \text{ and } \models \theta(R, c) \rightarrow (\phi(R, S') \rightarrow S'(c)).$$

Then $\phi(R, S) \models \forall x (S(x) \leftrightarrow \theta(R, x)).$
Example

Interpolation fails in finite models. Suppose $\phi$ is as follows:

$$\forall x \exists y S(x, y) \land$$
$$\forall x \forall y \forall y'((S(x, y) \land S(x, y')) \rightarrow y = y')$$
$$\forall x \forall y (S(x, y) \rightarrow (\neg x = y \land S(y, x)))$$

Suppose $\psi$ is

$$\exists z [S(z, z) \land \forall x \exists y S(x, y) \land$$
$$\forall x \forall y \forall y'((S(x, y) \land S(x, y')) \rightarrow y = y')$$
$$\forall x \forall y((S(x, y) \land \neg x = z) \rightarrow (\neg x = y \land S(y, x))))$$

Then $\models \phi \rightarrow \neg \psi$ but if $\theta$ is an interpolant, then $\theta$ is an identity-sentence which is true in exactly the finite models with even cardinality, which is impossible.
Remark

Single sorted logic second order logic satisfies trivially the Interpolation Theorem: Suppose $\models \phi \rightarrow \psi$, where $\phi$ is an $L_1$-sentence and $\psi$ is an $L_2$-sentence of second order logic. Then there is an $L_1 \cap L_2$-sentence $\theta$ of second order logic such that $\models \phi \rightarrow \theta$ and $\models \theta \rightarrow \psi$, obtained as follows. Let $L_1 \setminus L_2$ consist (only) of the relation symbols $R_1, \ldots, R_n$, $a(R_i) = m_i$. We can write $\phi$ as $\phi(R_1, \ldots, R_n)$. Let

$$\theta : \exists X^{m_1} \ldots \exists X^{m_n} \phi(X^{m_1}, \ldots, X^{m_n}).$$

Many sorted second order logic does *not* satisfy Interpolation Theorem (see later), but “sort logic” does.
Towards many sorted interpolation
Let $\text{Un}(\phi)$ be all sorts $s$ such that a variable of sort $s$ occurs universally quantified in $\phi$. Similarly $\text{Un}(S)$ for a set $S$ of formulas.

Let $\text{Ex}(\phi)$ be all sorts $s$ such that a variable of sort $s$ occurs existentially quantified in $\phi$. Similarly $\text{Ex}(S)$ for a set $S$ of formulas.

**Example**

- Suppose $\phi$ is $\forall x^1 \exists x^0 (x^0 = x^1)$. Then $\text{Un}(\phi) = \{1\}$ and $\text{Ex}(\phi) = \{0\}$.

- Suppose $\phi$ is $\forall x^0 \forall y^3 (R(x^0, y^3) \leftrightarrow R'(x^0, y^3))$. Then $\text{Un}(\phi) = \{0, 3\}$ and $\text{Ex}(\phi) = \emptyset$. 
Many Sorted Interpolation Theorem for $L_{\omega_1\omega}$ (or $L_{\omega\omega}$)

**Theorem**

We assume that $L_1$ and $L_2$ are relational. Suppose $\models \phi \rightarrow \psi$, where $\phi$ is an $L_1$-sentence and $\psi$ is an $L_2$-sentence of $L_{\omega_1\omega}$. Then there is an $L_1 \cap L_2$-sentence $\theta$ of $L_{\omega_1\omega}$ such that

1. $\models \phi \rightarrow \theta$
2. $\models \theta \rightarrow \psi$
3. $\text{Un}(\theta) \subseteq \text{Un}(\phi)$
4. $\text{Ex}(\theta) \subseteq \text{Ex}(\psi)$.

If $\phi$ and $\psi$ are strict, then so is $\theta$. 
Proof:

Let us assume that the claim of the theorem is false and derive a contradiction. Since $\models \phi \rightarrow \psi$, the set $\{\phi, \neg \psi\}$ has no models. We construct a consistency property for $\{\phi, \neg \psi\}$.

Let $L = L_1 \cap L_2$. Suppose $C^s = \{c^s_n : n \in \mathbb{N}\}$ is a set of new constant symbols for each sort $s$ of $L_1 \cup L_2$. Let $C^* = \bigcup_s C^s$.

Given a set $S$ of sentences, let $S_1$ consists of all $L_1 \cup C^*$-sentences in $S$ with only finitely many constant from $C^*$, and let $S_2$ consists of all $L_2 \cup C^*$-sentences in $S$ with only finitely many constant from $C^*$. 
Let us say that $\theta$ separates $S'$ and $S''$ if

1. $S' \models \theta$,
2. $S'' \models \neg \theta$,
3. $\text{Un}'(\theta) \subseteq \text{Un}(S')$,
4. $\text{Ex}'(\theta) \subseteq \text{Un}(S'')$,

where $\text{Un}'(\theta)$ consists of sorts $s \in \text{Un}(\theta)$ and sorts of constants $c \in C^*$ occurring in $\theta$, and $\text{Ex}'(\theta)$ consists of sorts $s \in \text{Ex}(\theta)$ and sorts of constants $c \in C^*$ occurring in $\theta$. 
Definition

Let $\Delta$ consist of finite sets $S$ of sentences of $L_{\omega_1 \omega}$ such that $S = S_1 \cup S_2$ and:

(⋆) There is no $L \cup C^*$-sentence that separates $S_1$ and $S_2$.

In the strict case we demand that all the sentences are strict.
\( \Delta \) is a consistency property

**Case 0.** \( \{ \phi, \neg \psi \} \in \Delta \). True by assumption. (We may assume that \( \phi \) is consistent and that \( \psi \) is not valid.)

**Case 1.** Suppose \( S \in \Delta \) and consider \( c = c \), where, for example, \( c \in L_1 \cup C^* \). We let \( S'_1 = S_1 \cup \{ c = c \} \) and \( S'_2 = S_2 \cup \{ c = c \} \). Suppose \( \theta(c_0, \ldots, c_{m-1}) \) separates \( S'_1 \) and \( S'_2 \). Then clearly also \( \theta(c_0, \ldots, c_{m-1}) \) separates \( S_1 \) and \( S_2 \), a contradiction.

**Case 2.** Suppose \( \phi(c) \in S \in \Delta \), \( \phi(c) \) atomic, and \( c = c' \in S \). Clearly \( S \cup \{ \phi(c') \} \in \Delta \).

**Case 3.** Negation: trivial.
Case 4. Consider $\bigvee_{i \in I} \varphi_i$, where for example $\bigvee_{i \in I} \varphi_i \in S_1$. We claim that for some $i \in I$ the sets $S_1 \cup \{\varphi_i\}$ and $S_2$ satisfy ($\ast$). Otherwise there is for each $i \in I$ some $\theta_i$ that separates $S_1 \cup \{\varphi_i\}$ and $S_2$. Let $\theta = \bigvee_{i \in I} \theta_i$. Then $\theta$ separates $S_1$ and $S_2$ contrary to assumption.

Case 5. Consider $\varphi_i$ where for example $\bigwedge_{i \in I} \varphi_i \in S_1$. Let $S'_1 = S_1 \cup \{\varphi_i\}$ and $S'_2 = S_2$. If $\theta$ separates $S'_1$ and $S'_2$, then clearly $\theta$ also separates $S_1$ and $S_2$. 
Case 6. Consider \( S \in \Delta \) and \( \exists x^S \phi(x^S) \in S_1 \). Let \( c_0 \in C^S \) be such that \( c \) does not occur in \( S \). We claim that the sets \( S_1 \cup \{ \phi(c_0) \} \) and \( S_2 \) satisfy \((\ast)\). Otherwise there is some \( \theta(c_0, \ldots, c_{m-1}) \) that separates \( S_1 \cup \{ \phi(c_0) \} \) and \( S_2 \). Let\(^1 \) \( \theta'(c_1, \ldots, c_{m-1}) = \exists x^S \theta(x^S, c_1, \ldots, c_{m-1}) \).

We show that \( \theta'(c_1, \ldots, c_{m-1}) \) separates \( S_1 \) and \( S_2 \), a contradiction. Checking this:

- \( S_1 \models \theta'(c_1, \ldots, c_{m-1}) \):

\[
S_1 \cup \{ \phi(c_0) \} \models \theta(c_0, \ldots, c_{m-1}) \text{ by assumption} \\
S_1 \models \phi(c_0) \rightarrow \theta(c_0, \ldots, c_{m-1}) \\
S_1 \models \forall x^S (\phi(x^S) \rightarrow \theta(x^S, c_1, \ldots, c_{m-1})) \\
S_1 \models \exists x^S \phi(x^S) \rightarrow \exists x^S \theta(x^S, c_1, \ldots, c_{m-1}) \\
S_1 \models \exists x^S \theta(x^S, c_1, \ldots, c_{m-1}) \text{ as } S_1 \models \exists x^S \phi(x^S) \\
S_1 \models \theta'(c_1, \ldots, c_{m-1})
\]

\(^1\)If \( c_0 \) does not occur in \( \theta \), then we take \( \theta' = \theta \).
Case 6. (Contd.)

\[ S_2 \models \neg \theta'(c_0, \ldots, c_{m-1}) : \]

\[ S_2 \models \neg \theta(c_0, \ldots, c_{m-1}) \]

\[ S_2 \models \forall x^s \neg \theta(x^s, c_1, \ldots, c_{m-1}) \]

\[ S_2 \models \neg \exists x^s \theta(x^s, c_1, \ldots, c_{m-1}) \]

\[ S_2 \models \neg \theta'(c_1, \ldots, c_{m-1}) \]
Case 6. (Contd.)

- **Un′(θ′(c₁, ..., cₘ₋₁)) ⊆ Un(S₁):**
  
  \[ s' \in \text{Un}'(\theta'(c₁, ..., cₘ₋₁)) \]
  
  \[ s' \in \text{Un}'(\theta(c₀, ..., cₘ₋₁)) \]
  
  \[ s' \in \text{Un}(S₁ \cup \{\phi(c₀)\}). \]

  If \( s' \in \text{Un}(\phi(c₀)) \), then \( s' \in \text{Un}(\exists x^s\phi(x^s)) \), whence \( s' \in \text{Un}(S₁) \).

- **Ex′(θ′(c₁, ..., cₘ₋₁)) ⊆ Un(S₂):**
  
  \[ s' \in \text{Ex}'(\theta'(c₁, ..., cₘ₋₁)) \]
  
  \[ s' \in \text{Ex}'(\theta(c₀, ..., cₘ₋₁)) \cup \{s\} \]
  
  \[ s' \in \text{Un}(S₂) \cup \{s\} = \text{Un}(S₂), \]

  since \( c₀ \) occurs in \( \theta(c₀, ..., cₘ₋₁) \), and hence \( c₀ \in \text{Un}(S₂) \) by the choice of \( \theta \).
Case 6. (Contd.) Consider $S \in \Delta$ and $\exists x^S \phi(x^S) \in S_2$. Let $c_0 \in C^s$ be such that $c_0$ does not occur in $S$. We claim that the sets $S_1$ and $S_2 \cup \{ \phi(c_0) \}$ satisfy $(\star)$. Otherwise there is some $\theta(c_0, \ldots, c_{m-1})$ that separates $S_1$ and $S_2 \cup \{ \phi(c_0) \}$. Let $\theta'(c_1, \ldots, c_{m-1}) = \forall x^s \theta(x^s, c_1, \ldots, c_{m-1})$. We show that $\theta'(c_1, \ldots, c_{m-1})$ separates $S_1$ and $S_2$, a contradiction. Checking this:

- $S_1 \models \theta'(c_1, \ldots, c_{m-1})$:

  $S_1 \models \theta(c_0, \ldots, c_{m-1})$

  $S_1 \models \forall x^s \theta(x^s, c_1, \ldots, c_{m-1})$

  $S_1 \models \theta'(c_1, \ldots, c_{m-1})$

---

If $c_0$ does not occur in $\theta$, we choose $\theta' = \theta$. 
\( S_2 \models \neg \theta'(c_1, \ldots, c_{m-1}) : \)

\[
S_2 \cup \{ \phi(c_0) \} \models \neg \theta(c_0, \ldots, c_{m-1})
\]

\( S_2 \models \phi(c_0) \rightarrow \neg \theta(c_0, \ldots, c_{m-1}) \)

\( S_2 \models \forall x^s(\phi(x^s) \rightarrow \neg \theta(x^s, c_1, \ldots, c_{m-1})) \)

\( S_2 \models \exists x^s \phi(x^s) \rightarrow \exists x^s \neg \theta(x^s, c_1, \ldots, c_{m-1}) \)

\( S_2 \models \exists x^s \neg \theta(x^s, c_1, \ldots, c_{m-1}) \)

\( S_2 \models \neg \theta'(c_1, \ldots, c_{m-1}) \)

\( \text{Un}'(\theta'(c_1, \ldots, c_{m-1})) \subseteq \text{Un}(S_1) : \)

\( s' \in \text{Un}'(\theta'(c_1, \ldots, c_{m-1})) \)

\( s' \in \text{Un}'(\theta(c_0, \ldots, c_{m-1}))^3 \)

\( s' \in \text{Un}(S_1) \) by the choice of \( \theta \).

\(^3\text{Remember that } c_0 \text{ occurs in } \theta.\)
\[ \text{Ex}'(\theta'(c_1, \ldots, c_{m-1})) \subseteq \text{Un}(S_2): \text{Suppose} \]

\[ s' \in \text{Ex}'(\theta'(c_1, \ldots, c_{m-1})). \]

Then \( s' \in \text{Ex}'(\theta(c_0, \ldots, c_{m-1})) \) because the universal quantifier in front of \( \theta' \) adds no new existentially quantified variables. By our assumption about \( \theta \), \( s' \in \text{Un}(S_2 \cup \{\phi(c_0)\}) \). If \( s' \in \text{Un}(\phi(c_0)) \), then \( s' \in \text{Un}(S_2) \). So in any case \( s' \in \text{Un}(S_2) \).
Case 7. Consider $S \in \Delta$, $\phi(c_0)$, where $c_0 \in C^s$ and $\forall x^s \phi(x^s) \in S_1$. Exercise!

Case 7. (Contd.) Consider $\phi(c_0)$, where $c_0 \in C^s$ and $\forall x^s \phi(x^s) \in S_2$. Exercise!
Applications of interpolation
Single sorted interpolation follows a fortiori.
Proposition

Suppose $\phi$ depends only on sort $s$ in the sense that

$$\mathcal{M} \models \phi \iff \mathcal{N} \models \phi$$

whenever $\mathcal{M}$ and $\mathcal{N}$ agree on sort $s$ universe, constants and relations.

Then $\models \phi \leftrightarrow \psi$ where only sort $s$ variables occur in $\psi$.

Proof.

Let $s_1, \ldots, s_n$ be the sorts in $\phi$ or $\psi$ in addition to $s$. Let $\phi'$ be the result of replacing each sort $s_i$ in $\phi$ by a new sort $s'_i$, systematically. Now

$$\models \phi \rightarrow \phi'.$$

By the Many Sorted Interpolation Theorem there is $\theta$ containing only sort $s$ such that $\models \phi \rightarrow \theta$ and $\models \theta \rightarrow \phi'$. Hence $\models \phi \leftrightarrow \theta$. 

\hfill \square
Many sorted interpolation gives preservation results.
Theorem (Łoś-Tarski-Malitz)

A formula $\phi$ is preserved by submodels if and only if it is logically equivalent to a universal formula. Even in $L_{\omega_1\omega}$.

Definition

Let $EXT$ be the conjunction of

$$\forall x^1 \exists x^0 (x^0 = x^1)$$

$$\forall x^1 \forall y^1 (R'(x^1, y^1) \leftrightarrow R(x^1, y^1))$$

This is true in $(\{M_0, M_1\}, R, R')$ iff $(M_1, R') \subseteq (M_0, R)$. 
Theorem (Łoś-Tarski-Malitz)

A formula $\phi$ is preserved by submodels if and only if it is logically equivalent to a universal formula. Even in $L_{\omega_1\omega}$.

Definition

Let $EXT$ be the conjunction of

$$\forall x^1 \exists x^0 (x^0 = x^1)$$

$$\forall x^1 \forall y^1 (R'(x^1, y^1) \leftrightarrow R(x^1, y^1))$$

This is true in $(\{M_0, M_1\}, R, R')$ iff $(M_1, R') \subseteq (M_0, R)$. 
Proof.
Let us assume the single sorted $\phi$ is written in sort 0 variables. Let $\phi'$ be the same written in sort 1 variables and with $R$ replaced by $R'$. Then

$$\text{EXT} \land \neg \phi' \models \neg \phi$$

$$\text{EXT} \land \neg \phi' \models \theta, \theta \models \neg \phi$$

The only common sort is 0, so $\theta$ has only sort 0 symbols. To see that $\theta$ is existential we note that if a sort 0 variable was universally quantified in $\theta$, then it is universally quantified in $\text{EXT} \land \neg \phi'$, but there is no universally quantified sort 0 variable in $\text{EXT} \land \neg \phi'$. Thus $\theta$ is existential. Moreover, $\models \neg \phi \leftrightarrow \theta$. 
Definition

A flow formula is a formula of the vocabulary of Chu spaces made of atomic formulas and their negations in the lax 2-sorted logic of Chu spaces by means of $\land, \lor, \exists x^0, \forall x^1$.

Definition

A Chu transform is simple if its mappings $g$ and $h$ are identities.

Lemma

Flow formulas are preserved by simple Chu transforms.

Proof.

Easy induction on formulas.
Theorem (van Benthem)

A formula of 2-sorted first order logic of Chu spaces is preserved by simple Chu transforms if and only if it is equivalent to a flow-formula.
Suppose $\phi$, written in sorts 0 and 1, is preserved by simple Chu transforms. Then $\phi$ is existential in sort 0: Let $\phi'$ be $\phi$ but in sort 2 and 3, and $R'$ instead of $R$. Let

$$EXT_1 : \forall x^0 \exists x^2 x^0 = x^2$$

$$EXT_2 : \forall x^0 \forall y^3 (R(x^0, y^3) \leftrightarrow R'(x^0, y^3))$$

$$EXT_3 : \forall x^3 \exists x^1 x^3 = x^1$$

$EXT_1 \land EXT_2 \land EXT_3$ is true in $(\{M_0, M_1, M_2, M_3\}, R, R')$ iff id is a simple Chu transform $(M_0, M_1, R) \rightarrow (M_2, M_3, R')$.

Now

$$\phi \land EXT_1 \land EXT_2 \land EXT_3 \models \phi'.$$

We need and interpolant $\theta'$ in sorts 2 and 3 such that sort 2 is existential only in $\theta'$, so on the left hand side we have arranged to have no universal sort 2. Interpolation gives $\theta'$ with

$$\phi \land EXT_1 \land EXT_2 \land EXT_3 \models \theta', \theta' \models \phi'.$$

Now $\theta'$ is in sorts 2 and 3 and existential in sort 2. Clearly $\models \phi \leftrightarrow \theta$ if we get $\theta$ from $\theta'$ by switching back to sort 0 and 1.
So we have \( \phi \), existential in sort 0 which is preserved by simple Chu transforms. We need \( \phi \) to be universal in sort 1. We have

\[
\phi \models (\text{EXT}_1 \land \text{EXT}_2 \land \text{EXT}_3) \rightarrow \phi'.
\]

We need an interpolant \( \theta' \) in sorts 0 and 1 so that sort 1 is universal only in \( \theta' \), but fortunately on the right hand side we have no existential sort 1. Interpolation gives \( \theta' \) such that

\[
\phi \models \theta', \theta' \models (\text{EXT}_1 \land \text{EXT}_2 \land \text{EXT}_3) \rightarrow \phi'.
\]

Now \( \theta' \) is in sorts 0 and 1 and universal in sort 1. Moreover it is existential in 0, because \( \phi \) is. Clearly \( \models \phi \leftrightarrow \theta \) if we get \( \theta \) from \( \theta' \) by switching back to \( R \).
Between set theory and model theory.
The Härtig-quantifier is the generalised quantifier

\[ lxy \phi(x) \psi(y) \iff |\phi(\cdot)| = |\psi(\cdot)|. \]

The extension of first order logic by \( l \) is denoted \( L(l) \).

Theorem (Lindström)

*The class of well-orders \((M, <)\) is the class of reducts of models of a sentence of \( L(l) \). So is the class of linear orders which are not well-orders.*
Proof.

Let 1 be a new sort and $R$ a new predicate of sort $\langle 0, 1 \rangle$. Let $\theta$ be the conjunction of:

$$\forall x^0, y^0 (x^0 < y^0 \rightarrow \forall x^1 (R(x^0, x^1) \rightarrow R(y^0, x^1)))$$

$$\forall x^0, y^0 (x^0 < y^0 \rightarrow \neg \exists x^1 y^1 R(x^0, x^1)R(y^0, y^1))$$

A linear order $(M, <)$ is a well order iff $(\{M\}, <)$ is a reduct of a model $(\{M, N\}, <, R)$.

Note: The class of single-sorted well-orders $(M, <)$ is not the class of reducts of models of a single-sorted sentence of $L(I)$. The reason: Consider countable models. Here $I$ would be eliminable if $L_{\omega_1 \omega}$ is used. But well-order is not definable in $L_{\omega_1 \omega}$ even in countable models.
What is the use of being able to express well-order?

Lemma (Mostowski’s Collapsing Lemma)

If \( M \) is a model in which \((M_s, E) \models \text{Axiom of Extensionality}\) and \((M_s, E)\) is well-founded, then there is \( M' \cong M \) such that \( M'_s \) is a transitive set and \( E' \) is the membership relation \((\in)\) restricted to \( M'_s \).

Proof.

Define for \( x \in M_s \):

\[
\pi(x) = \{ \pi(y) : y \in M_s, yEx \}.
\]

Since \( E \) is well-founded, this is a legitimate definition by transfinite recursion. The function \( \pi \) is one-one, because \((M, E)\) satisfies the Axiom of Extensionality. Let \( M' \cong M \) so that \( \pi : (M_s, E) \cong (M'_s, \in) \).

\( \square \)
Definition

A model \((M', E')\), \(E' \subseteq M'^2\), is an end-extension of \((M, E)\), \(E \subseteq M^2\), in symbols \((M, E) \subseteq_{end} (M', E')\), if \((M, E) \subseteq (M', E')\) and
\[
\forall x \in M \forall x' \in M' (x' E' x \rightarrow x' \in M).
\]

Remark

A special case is a transitive set which is any set \(M\) such that \((M, \in) \subseteq_{end} (V, \in)\).
Definition

A formula of set theory is \textit{bounded} if all of its quantifiers are bounded, i.e. of the form $\forall x \in y$ or $\exists x \in y$.

Definition

A class $C$ (of set theory) is $\Sigma_1$ (-definable) if there is a bounded formula $\psi(x, y)$ such that for all $a$:

$$a \in C \iff \exists y \psi(\bar{s}, y).$$

We say that a formula $\phi(\bar{x})$ is $\Sigma_1^V$ (sometimes we omit "$V$") if the class it defines is. If the equivalence is provable in a theory $T$, then $\phi(\bar{x})$ is said to be $\Sigma_1^T$. Similarly $\Pi_1^V$, $\Pi_1^T$, $\Delta_1^V$, and $\Delta_1^T$.

We are particularly interested in classes that are model classes, i.e. classes of models of the same vocabulary, closed under isomorphisms.
An application of many-sorted interpolation

Theorem (Feferman-Kreisel)

A formula of set theory is preserved by end-extensions of models of ZFC if and only if it is $\Sigma^\text{ZFC}_1$. (ZFC can be weakened here.)

Proof.

First one proves that formulas are equivalent in ZFC to $\Sigma_1$-formulas are preserved in end-extensions. This is easy. Then one has to check the proof of the many-sorted interpolation theorem to see that it works if bounded quantifiers are not counted when computing $\text{Un}(\phi)$ and $\text{Ex}(\phi)$. The rest is like the proof of Łoś-Tarski Preservation Theorem.
Corollary

If a formula $\phi(x)$ of the language $\{\in\}$ (sort $s$) of set theory is $\Sigma_1$ in sort $s$ and

$$\mathcal{M} \models \phi(n),$$

where $\mathcal{M}$ is such that $(M_s, E) \models \text{ZFC}_n$ (a large finite fragment of ZFC) and $(M_s, E)$ is well-founded, then $\phi(n)$ is true.

Note: "well-founded" can be omitted if $\phi$ is $\Sigma_1^{KP}$, where $KP$ is the Kripke-Platek set theory.

Proof.

Let $\mathcal{M}' \cong \mathcal{M}$ such that $M'_s$ is transitive and $E' = \in$. Then $(M'_s, \in) \models \phi(n)$. Since $M'_s$ is transitive, $(M'_s, \in) \subseteq_{\text{end}} (V, \in)$, so $\phi(n)$ is actually true.
Transitive closure

**Lemma**

*Every set x is element in a smallest transitive set, called the transitive closure of x, in symbols TC(x).*

**Proof.**

Let \( TC_0(x) = \{x\} \), \( TC_{n+1}(x) = TC_n(x) \cup \{y : \exists x \in TC_n(x)(y \in x)\} \), \( TC(x) = \bigcup_n TC_n(x) \).
Definition

The set of sets of hereditary cardinality $< \kappa$, $H_\kappa$, is the set of sets $x$ such that $|TC(x)| < \kappa$.

Remark

It is easy to see that for regular $\kappa$, $(H_\kappa, \in) \models ZFC^-.$

Theorem (Levy Reflection Principle)

If $\kappa$ is regular, then $(H_\kappa, \in) \prec_1 (V, \in)$ i.e. if $\phi(x)$ is $\Sigma_1^{ZFC^-}$ and $a \in H_\kappa$, then

$$H_\kappa \models \phi(a) \iff V \models \phi(a).$$

Proof.

To be given on the blackboard.
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\[ H_\kappa \models \phi(a) \iff V \models \phi(a). \]

**Proof.**

To be given on the blackboard.
Theorem

Every $\Sigma^V_1$-definable (in set theory) model class is the class of reducts of models of a sentence of $L(I)$.

Proof

Suppose $K$ is defined by the $\Sigma_1$-formula $\phi(x)$. We assume the vocabulary $\{R\}$, $R$ binary, is (single sorted and) in sort 0. Let $E$ be a binary predicate of sort 1, $f$ a new function symbol of sort $\langle 0, 1 \rangle$, and $c$ a new constant of sort 1. Let $\theta \in L(I)$ so that the reducts of models of $\theta$ in sort 1 and predicate $E$ are exactly the well-founded binary structures. We can assume that sort 0 does not occur in $\theta$. Let $\Phi$ be the conjunction of:
“$(M_1, E) \models ZFC$”

θ (to make sure we deal with a well-founded model)

$\phi(c)$

“$c \in M_1$ is an $L$-structure $(U^c, P^c)_E$" (i.e. in the sense of $E$).

“$f$ is an isomorphism between sort 0 and $c$" (e.g.

$\forall x^0 \forall y^0 (f(x^0) = f(y^0) \to x^0 = y^0), \forall x^0 (f(x^0) E U^c)$ and

$\forall x^0 \forall y^0 (R(x^0, y^0) \leftrightarrow \langle f(x^0), f(y^0) \rangle_E E E P^c$, etc.)

Claim: $\mathcal{A} \in K$ if and only if $\mathcal{A}$ is the 0-reduct of a model of $\Phi$.

Left to right: Suppose $\mathcal{A} \in K$, i.e. $V \models \phi(\mathcal{A})$. Pick $\kappa$ so that $\mathcal{A} \in H_\kappa$. By Levy Reflection, $H_\kappa \models \phi(\mathcal{A})$. By using $(H_\kappa, \in)$ for sort 1 we can easily find $\mathcal{M} \models \Phi$ such that $(M_0, R) = \mathcal{A}$. 
Claim: $A \in K$ if and only if $A$ is the 0-reduct of a model of $\Phi$.

Right to Left: Suppose $\mathcal{M} \models \Phi$ and $A = (M_0, R)$. Let $\mathcal{M}' \cong \mathcal{M}$ such that $(M'_1, E')$ is a transitive $\epsilon$-structure $(M'_1, \in)$. Since $K$ is closed under $\cong$, it suffices to show that $\phi(A')$, where $A' = (M'_0, R')$, is true. Since $M'_1$ is a transitive model of $ZFC^\neg$, the set $c^{M'}$ is a real $L$-structure $\mathcal{B}$, thanks to $f^{\mathcal{M}'}$, $\mathcal{B} \cong A'$. So it suffices to prove $\phi(\mathcal{B})$. We know that $(M'_1, \in) \models \phi(\mathcal{B})$. Since $\phi(x)$ is $\Sigma_1$, we get $V \models \phi(\mathcal{B})$, as desired.
Definition

Let $Cd(x)$ be the predicate “$x$ is a cardinal number” of set theory. Let $Pw(x, y)$ be the predicate “$x$ is the power-set of $y$” of set theory. We use $\Sigma_1(Cd)^V$ to denote the set of formulas which are $\Sigma_1^V$ in the extended language $\{\in, Cd\}$, respectively $\Sigma_1(Pw)^V$, $\Sigma_1(Pw)^T$, etc.

Remark

- Both $Cd$ and $Pw$ are $\Pi_1$-predicates.
- $\Sigma_1(Cd)^{ZFC} \subseteq \Sigma_1(Pw)^{ZFC} = \Sigma_2^{ZFC}$
- $\Sigma_1(Cd)^{ZFC+V=L} = \Sigma_1(Pw)^{ZFC+V=L}$.
- If a model class is $\Sigma_1(Cd)^T$ and $T$ is true (i.e. assumed), then the model class is $\Sigma_1(Cd)^V$.
- Much more model classes are $\Sigma_1(Cd)^{ZFC}$ than are merely $\Sigma_1^{ZFC}$
- The following classes are $\Sigma_1(Cd)^{ZFC}$:
Remark

The following classes are $\Sigma_1(Cd)^{ZFC}$: The class of $(M, P)$, $P \subseteq M$, where

- $|P|$ is finite.
- $|P|$ is finite and even.
- $|P|$ is the code of a Turing machine that halts.
- $|P|$ is the Gödel number of a first order sentence true in $(\mathbb{N}, +, \cdot, 0, 1)$.
- $|P|$ is countable.
- $|P| = \aleph_n$ (any fixed $n$).
- $|P| = \aleph_n$ for some $n$.
- $|P| = \aleph_{|P|}$. 
Remark

- \((H_\kappa, \in) \models Cd(\alpha) \text{ if and only if } \alpha \text{ is a real cardinal.}\)
- We say that a transitive \((M, \in)\) is Cd-absolute if for all \(\alpha \in M:\)
  \((M, \in) \models Cd(\alpha) \text{ if and only if } \alpha \text{ is a real cardinal.}\)
- If \((M, \in)\) is Cd-absolute, then for \(\Sigma_0\)-formulas of the vocabulary \(\{\in, Cd\}\) we have \((M, \in) \models \phi(a) \text{ if and only if } \phi(a) \text{ is true. In other words } (M, \in) \prec_{\Sigma_0(Cd)} (V, \in)\)
- “x is transitive” is \(\Delta^ZFC_1\)
- “x is Cd-absolute” is \(\Delta_1(Cd)^{ZFC}\)
Theorem

Every $\Sigma_1^Y(Cd)$-definable (in set theory) model class is the class of reducts of models of a sentence of $L(I)$.

Proof

Suppose $K$ is defined by the $\Sigma_1(Cd)$-formula $\phi(x)$. We assume the vocabulary $\{R\}$, $R$ binary, is (single sorted and) in sort 0. Let $E$ be a binary predicate of sort 1, $f$ a new function symbol of sort $\langle 0, 1 \rangle$, and $c$ a new constant of sort 1. Let $\theta \in L(I)$ so that the reducts of models of $\theta$ in sort 1 and predicate $E$ are exactly the well-founded binary structures. We can assume that sort 0 does not occur in $\theta$. Let $\Phi$ be the conjunction of:
“(M₁, E) ⊨ ZFC−”

(1) \( \theta \) (to make sure we deal with a well-founded model)

(2) \( \phi(c) \)

(3) “c ∈ M₁ is an L-structure \((U^c, P^c)_E\)” (i.e. in the sense of E).

(4) “f is an isomorphism between sort 0 and c” (e.g.
   \[ \forall x^0 \forall y^0 (f(x^0) = f(y^0) \rightarrow x^0 = y^0), \forall x^0 (f(x^0) \in U^c) \text{ and} \]
   \[ \forall x^0 \forall y^0 (R(x^0, y^0) \leftrightarrow \langle f(x^0), f(y^0) \rangle_E \in E P^c), \text{etc.} \]

(5) “Cardinals of \((M₁, E)\) are real cardinals” i.e.
   \[ \forall x^1 (x^1 \text{ is a cardinal}_E \rightarrow \forall y^1 (y^1 \in x^1 \rightarrow \neg \exists z^1 v^1 (z^1 \in E y^1 ) (v^1 \in E x^1 )) \]

Claim: \( \mathcal{A} \in K \) if and only if \( \mathcal{A} \) is the 0-reduct of a model of \( \Phi \).

Left to right: Suppose \( \mathcal{A} \in K \), i.e. \( V \models \phi(\mathcal{A}) \). Pick \( \kappa \) so that \( \mathcal{A} \in H_\kappa \). By Levy Reflection, \( H_\kappa \models \phi(\mathcal{A}) \). By using \((H_\kappa, \in)\) for sort 1 we can easily find \( M \models \Phi \) such that \((M_0, R) = \mathcal{A}\).
Claim: $A \in K$ if and only if $A$ is the 0-reduct of a model of $\Phi$.

Right to Left: Suppose $\mathcal{M} \models \Phi$ and $A = (M_0, R)$. Let $\mathcal{M}' \cong \mathcal{M}$ such that $(M_1', E')$ is a transitive $\epsilon$-structure $(M_1', \in)$, which is $Cd$-absolute. Since $K$ is closed under $\cong$, it suffices to show that $\phi(A')$, where $A' = (M_0', R')$, is true. Since $M_1'$ is a transitive model of $ZFC^-$, the set $c^{M'}$ is a real $L$-structure $\mathcal{B}$, thanks to $f^{M'}$, $\mathcal{B} \cong A'$. So it suffices to prove $\phi(\mathcal{B})$. We know that $(M_1', \in) \models \phi(\mathcal{B})$. Since $\phi(x)$ is $\Sigma_1(Cd)$ and $(M_1', \in)$ is $Cd$-absolute, we get $V \models \phi(\mathcal{B})$, as desired.
Now the converse...

**Theorem**

*If a model class is the class of reducts of a sentence of $L(I)$, then it is $\Sigma_1(Cd)^V$-definable in set theory.*

Proof: Suppose $C$, a model class in a finite vocabulary $L = \{ R \}$, $R$ binary in sort 0, is the class of reducts to 0 of $\phi_0 \in L(I)$.

$$A \in C \iff \exists M (M \models \phi \land (A = (M_0, R))).$$

This already looks like $\Sigma_1$, but we have to work more.
Lemma

Suppose \((M, \in) \models ZFC^-\) is transitive and \(Cd\)-absolute. Then for all \(A \in M\) and \(\phi \in L(I)\):

\[A \models \phi \iff (M, \in) \models \text{“}A \models \phi\text{“}.\]

Proof.

By induction on \(\phi\).
Back to the proof of the theorem:

Let $\Phi(x, y, z)$ be the conjunction of the following formulas of set theory

1. $x$ is an $L$-structure,
2. $y$ is a structure,
3. $x$ is the reduct of $y$ to sort 0.
4. $z$ is a $Cd$-absolute transitive set, $x, y \in z$.
5. $((z, \in) \models "y \models \phi_0")$ in the language of set theory

Claim: $\mathcal{A} \in C \iff \exists y \exists z \Phi(\mathcal{A}, y, z)$.

Left to right: Suppose $\mathcal{A} \in C$. Thus $\mathcal{B} \models \phi_0$ for some $\mathcal{B}$ such that $(B_0, R) = \mathcal{A}$. Find $\kappa$ such that $\mathcal{B} \in H_\kappa$. Then $H_\kappa \models "\mathcal{B} \models \phi_0"$ since $H_\kappa$ is $Cd$-absolute. So we have $\Phi(\mathcal{A}, \mathcal{B}, H_\kappa)$.

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4 i.e. “$y \models \phi_0$” is written in the language of set theory and then $\forall$ and $\exists$ are relativized to $z$.
Right to left: Suppose $\Phi(\mathcal{A}, \mathcal{B}, N)$, where $N$ is a transitive $Cd$-absolute set such that

$N \models \text{"} \mathcal{B} \models \phi_0 \text{"}$.

Since $(N, \in) \prec_{\Sigma_1(Cd)} (V, \in)$, we have $\mathcal{B} \models \phi_0$, hence $\mathcal{A} \in C$. 
Corollary

TFAE:

1. \( C \) is the class of reducts of a sentence of \( L(I) \).
2. \( C \) is \( \Sigma_1^V(Cd) \)-definable in set theory

Corollary

TFAE:

1. \( C \) and the complement of \( C \) are classes of reducts of sentences of \( L(I) \).
2. \( C \) is \( \Delta_1^V(Cd) \)-definable in set theory
Corollary

TFAE:

1. $C$ is the class of reducts of a sentence of $L(I)$.
2. $C$ is $\Sigma_1^V(Cd)$-definable in set theory

Corollary

TFAE:

1. $C$ and the complement of $C$ are classes of reducts of sentences of $L(I)$.
2. $C$ is $\Delta_1^V(Cd)$-definable in set theory
**Definition**

A model class (a sentence) $K$ is $\Delta$-definable in a logic $L^*$, $K \in \Delta(L^*)$, if $K$ is the class of reducts of a sentence of $L^*$ and so is the complement of $K$.

**Corollary**

*TFAE:*

1. $C \in \Delta(L(I))$.
2. $C$ is $\Delta^V_1(Cd)$-definable in set theory
Similarly for second order logic $L^2$:

**Theorem**

*TFAE:*

1. $C \in \Delta(L^2)$.
2. $C$ is $\Delta_2^V$-definable in set theory.
Lemma

If $L^*$ satisfies the (many-sorted) interpolation theorem, then $\Delta(L^*) = L^*$.

Proof.

Suppose $K$ is the class of reducts of $\phi$ and the complement of $K$ is the class of reducts of $\psi$. Then $\models \phi \rightarrow \neg \psi$. Suppose $\theta$ is an interpolant. Then $K$ is the class of models of $\theta$. \qed
Example

The main (if not only) example known today:

1. $\Delta(L_{\omega\omega}) = L_{\omega\omega}$
2. $\Delta(L_{\omega_1\omega}) = L_{\omega_1\omega}$
3. $\Delta(L_A) = L_A$ for countable “admissible” fragments $L_A$ of $L_{\omega_1\omega}$.
4. $\Delta(L(Q_0)) = L_{\text{HYP}}$, where $\text{HYP}$ is the smallest admissible set.
5. $\Delta(L(Q_1))$ cannot be obtained from first order logic by adding finitely many generalised quantifiers (Shelah-V. to appear).
6. $\Delta(L(I)) \neq L(I)$, $\Delta(L^2) \neq L^2$. (An undefinability of truth argument à la Tarski.)
For many (most?) logics $L^*$ stronger than first order logic one can find a predicate $P$ of set theory so that definability in $\Delta(L^*)$ is equivalent to $\Delta_1(P)^V$-definability in set theory.

For first order logic use $\Delta_1^T$, where $T$ is Kripke-Platek set theory with urelements, minus Axiom of Infinity.

This is a **symbiosis** between model theory and set theory.