

Contextual Semantics:
From Quantum Mechanics to Logic, Databases,
Constraints, and Complexity
Lecture 1

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Beginnings ...

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It is no longer safe to assume this!

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- There is a fascinating two-way interplay developing between Computer Science and Physics, extending to the foundations of both, as well as to more practical matters. Quantum technology — “hacking matter” — will be a huge feature of 21st Century science and engineering, and a lot of it will be to do with information.

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- There is a fascinating two-way interplay developing between Computer Science and Physics, extending to the foundations of both, as well as to more practical matters. Quantum technology — “hacking matter” — will be a huge feature of 21st Century science and engineering, and a lot of it will be to do with information.
- This is an exciting emerging area, attracting students with backgrounds in CS, Physics, Mathematics, Philosophy, ...

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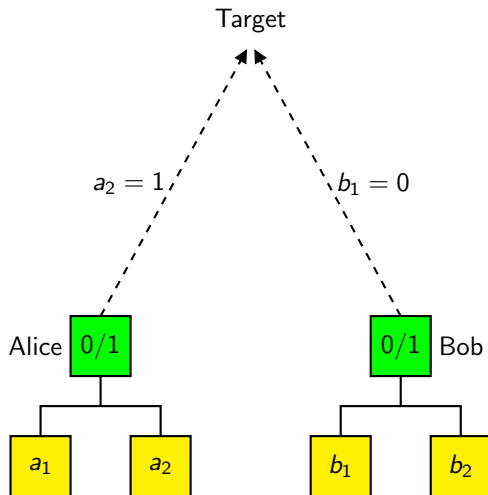
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- So, let's go right to the heart of the quantum mystery . . .

Alice and Bob look at bits



A Probabilistic Model Of An Experiment

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Example: The Bell Model

A	B	(0,0)	(1,0)	(0,1)	(1,1)
a_1	b_1	$1/2$	0	0	$1/2$
a_1	b_2	$3/8$	$1/8$	$1/8$	$3/8$
a_2	b_1	$3/8$	$1/8$	$1/8$	$3/8$
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The entry in row 2 column 3 says:

If Alice looks at a_1 and Bob looks at b_2 , then 1/8th of the time, Alice sees a 0 and Bob sees a 1.

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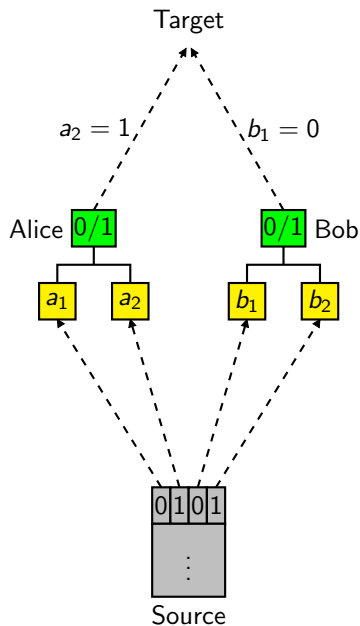
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If Alice looks at a_1 and Bob looks at b_2 , then $1/8$ th of the time, Alice sees a 0 and Bob sees a 1.

How can we explain this behaviour?

Classical Correlations



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Using elementary probability theory, we can calculate:

$$p_N \leq \text{Prob}\left(\bigvee_{i=1}^{N-1} \neg\phi_i\right) \leq \sum_{i=1}^{N-1} \text{Prob}(\neg\phi_i) = \sum_{i=1}^{N-1} (1 - p_i) = (N-1) - \sum_{i=1}^{N-1} p_i.$$

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Hence we obtain the inequality

$$\sum_{i=1}^N p_i \leq N - 1.$$

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$$\varphi_1 = a \wedge b \vee \neg a \wedge \neg b = a \leftrightarrow b$$

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The violation of the logical Bell inequality is 1/4.

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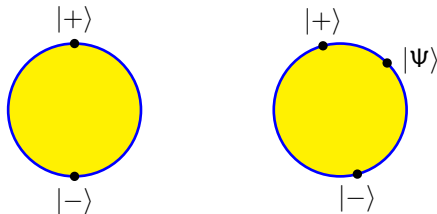
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Alice and Bob's choices are now of **measurement setting** (e.g. which direction to measure spin) rather than “which register to load”.

The Quantum Case: Spin Measurements

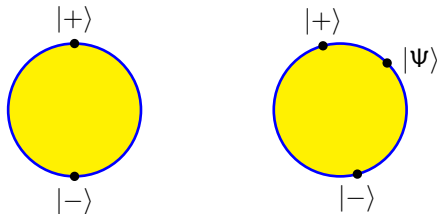
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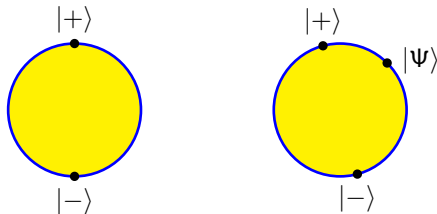
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Spin can be measured in any direction; so there are a continuum of possible measurements. There are **two possible outcomes** for each such measurement; spin in the specified direction, or in the opposite direction. These two directions are represented by a pair of orthogonal vectors. They are represented on the sphere as a pair of **antipodal points**.

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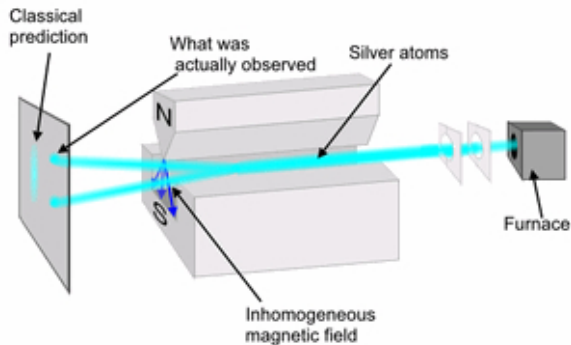
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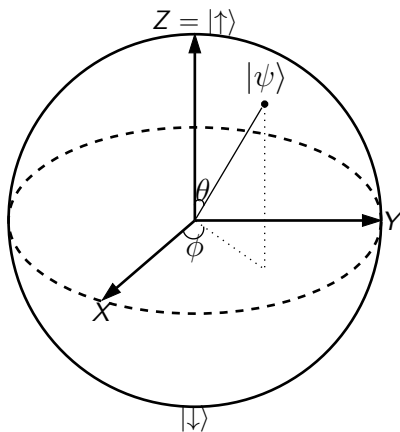
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Note the appearance of **quantization** here: there are not a continuum of possible outcomes for each measurement, but only two!

The Stern-Gerlach Experiment



The Bloch sphere representation of qubits



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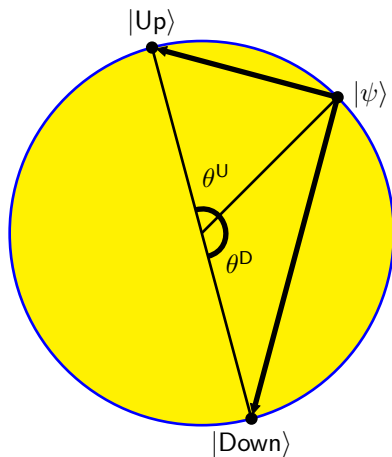
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- When we subject a qubit to a measurement (Up, Down), the state of the qubit determines a probability distribution on the two possible outcomes. The probabilities are determined by the **angles** between the qubit state $|\psi\rangle$ and the points ($|\text{Up}\rangle, |\text{Down}\rangle$) which specify the measurement. In algebraic terms, $|\psi\rangle, |\text{Up}\rangle$ and $|\text{Down}\rangle$ are unit vectors in the complex vector space \mathbb{C}^2 , and the probability of observing Up when in state $|\psi\rangle$ is given by the square modulus of the inner product:

$$|\langle\psi|\text{Up}\rangle|^2.$$

This is known as the **Born rule**. It gives the basic predictive content of quantum mechanics.

Truth makes an angle with reality



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But in addition, we have the important feature that there are a continuum of possible questions we can ask. However, note that on each run of the system, we can only ask **one** of these questions. We cannot simultaneously observe Up or Down in two different directions. Note that this corresponds to the feature of the scenario we discussed, that Alice and Bob could only look at one their local registers on each round.

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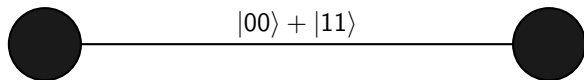
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Note in addition that a measurement has an **effect** on the state, which will no longer be the original state $|\psi\rangle$, but rather one of the states Up or Down, in accordance with the measured value.

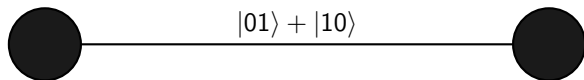
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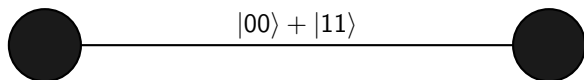


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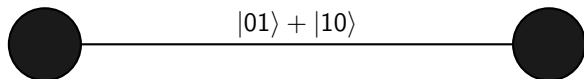


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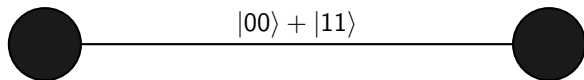
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$$\sum_i \lambda_i \cdot \phi_i \otimes \psi_i$$

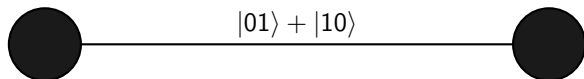
Superposition encodes **correlation**.

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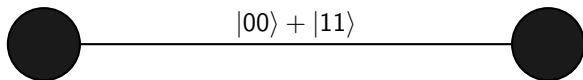
$$\sum_i \lambda_i \cdot \phi_i \otimes \psi_i$$

Superposition encodes **correlation**.

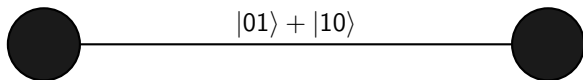
Einstein's 'spooky action at a distance'. Even if the particles are spatially separated, measuring one has an effect on the state of the other.

Quantum Entanglement

Bell state:



EPR state:



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Bell's theorem: QM is **essentially non-local**.

A Probabilistic Model Of An Experiment

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Example: The Bell Model

A	B	(0,0)	(1,0)	(0,1)	(1,1)
a_1	b_1	$1/2$	0	0	$1/2$
a_1	b_2	$3/8$	$1/8$	$1/8$	$3/8$
a_2	b_1	$3/8$	$1/8$	$1/8$	$3/8$
a_2	b_2	$1/8$	$3/8$	$3/8$	$1/8$

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Important note: this is **physically realizable!**

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Important note: this is **physically realizable!**

Generated by Bell state

$$\frac{|00\rangle + |11\rangle}{\sqrt{2}},$$

subjected to measurements in the XY -plane, at relative angle $\pi/3$.

A Probabilistic Model Of An Experiment

Example: The Bell Model

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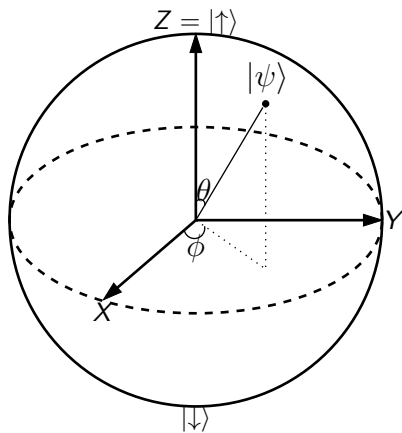
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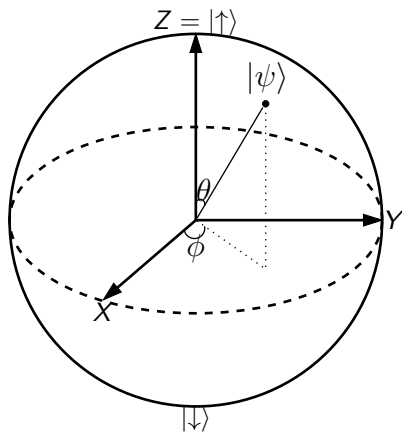
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Extensively tested experimentally.

Computing the Bell table

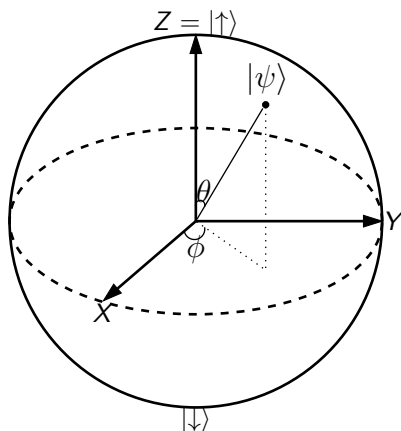


Computing the Bell table



Spin measurements lying in the equatorial plane of the Bloch sphere
Spin Up: $(|\uparrow\rangle + e^{i\phi}|\downarrow\rangle)/\sqrt{2}$, Spin Down: $(|\uparrow\rangle + e^{i(\phi+\pi)}|\downarrow\rangle)/\sqrt{2}$

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X itself, $\phi = 0$:

Spin Up $(|\uparrow\rangle + |\downarrow\rangle)/\sqrt{2}$ and Spin Down $(|\uparrow\rangle - |\downarrow\rangle)/\sqrt{2}$.

Computing the Bell table

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Alice: $a = X$, a' at $\phi = \pi/3$ (on **first** qubit)

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The event in yellow is represented by

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Probability of this event M when measuring (a, b') on $B = (|\uparrow\uparrow\rangle + |\downarrow\downarrow\rangle)/\sqrt{2}$ is given by Born rule:

$$|\langle B|M\rangle|^2.$$

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Since the vectors $|\uparrow\uparrow\rangle, |\uparrow\downarrow\rangle, |\downarrow\uparrow\rangle, |\downarrow\downarrow\rangle$ are pairwise orthogonal, $|\langle B|M\rangle|^2$ simplifies to

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The other entries can be computed similarly.