# Contextual Semantics: From Quantum Mechanics to Logic, Databases, Constraints, and Complexity Lecture 2

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# A Possibilistic Model Of An Experiment

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Can we explain this behaviour using a classical source?

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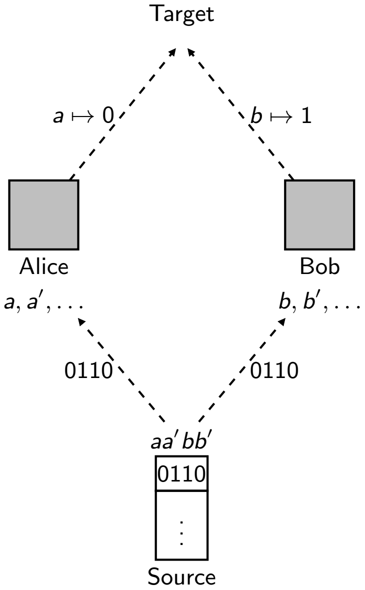
This point of view is called **non-contextuality**. It is equivalent to the assumption of a classical source.

However, this view is **impossible to sustain** in the light of our **actual observations of (micro)-physical reality**.

# Hidden Variables: The Mermin instruction set picture



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Thus Hardy models are **contextual**. They cannot be explained by a classical source.

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Each row of the table specifies a **probability distribution** on events $O^C$ for a given choice of measurements $C$.

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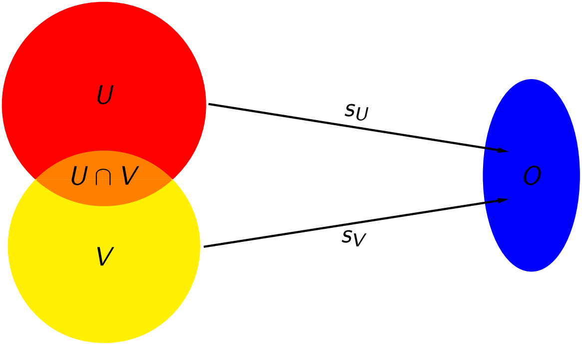
The different sets of compatible measurements correspond to the different contexts of measurement and observation of the physical system.

The fact that the behaviour of these observable outcomes cannot be accounted for by some context-independent global description of reality corresponds to the geometric fact that these local sections cannot be glued together into a **global section**.

# Gluing functional sections



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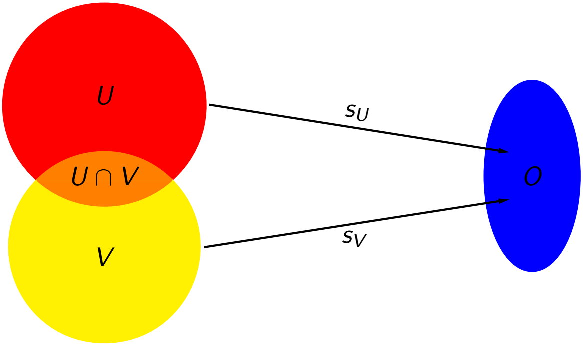


If $s_U|_{U\cap V} = s_V|_{U\cap V}$, they can be glued to form

$$s : U \cup V \longrightarrow O$$

such that $s|_U = s_U$ and $s|_V = s_V$.

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Examples: $\mathbb{R}_{\geq 0}$ (probability distributions), $\mathbb{B}$ (non-empty subsets), $\mathbb{R}$ (signed measures).

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$$\mathcal{D}_R(O^{U'}) \to \mathcal{D}_R(O^U) :: d \mapsto d|_U,$$

where for each $s : U \to O$:

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In fact, it is the composition of $U \mapsto O^U$ and the covariant distribution functor $\mathcal{D}_R$.

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**Example: Bell scenarios**
If $X_A$, $X_B$ are disjoint sets of labels for measurements by Alice and Bob, the set of contexts will be

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N.B. Vorob'ev theorem.

# Example: the 18-vector configuration in $\mathbb{R}^4$

This uses the following measurement cover $\mathcal{U} = \{U_1, \ldots, U_9\}$:

| $U_1$ | $U_2$ | $U_3$ | $U_4$ | $U_5$ | $U_6$ | $U_7$ | $U_8$ | $U_9$ |
|---|---|---|---|---|---|---|---|---|
| $A$ | $A$ | $H$ | $H$ | $B$ | $I$ | $P$ | $P$ | $Q$ |
| $B$ | $E$ | $I$ | $K$ | $E$ | $K$ | $Q$ | $R$ | $R$ |
| $C$ | $F$ | $C$ | $G$ | $M$ | $N$ | $D$ | $F$ | $M$ |
| $D$ | $G$ | $J$ | $L$ | $N$ | $O$ | $J$ | $L$ | $O$ |

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We can label these letters with vectors in $\mathbb{R}^4$ such that the vectors in each column are orthogonal.

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| $C$ | $F$ | $C$ | $G$ | $M$ | $N$ | $D$ | $F$ | $M$ |
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We can label these letters with vectors in $\mathbb{R}^4$ such that the vectors in each column are orthogonal.

Yields a proof of the **Kochen-Specker theorem**.

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**Contextual Probability Theory**!

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A setting for **contextual probability**.

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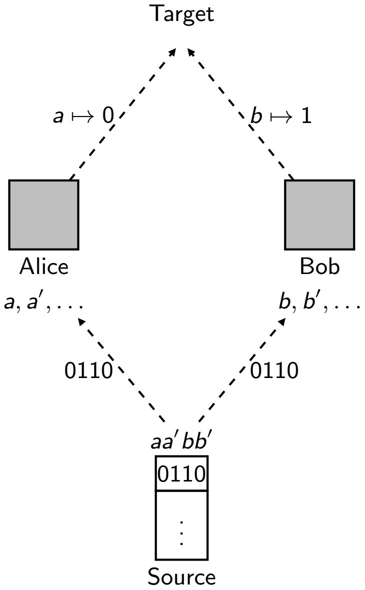
In other words, Bob's choice of measurement cannot influence Alice's outcome.

# Hidden Variables: The Mermin instruction set picture



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If $d$ is a global section for the model $\{e_C\}$, we recover the predictions of the model by **averaging over the values of these hidden variables**:

$$e_C(s) \ = \ d|C(s) \ = \sum_{s' \in \mathcal{E}(X), s'|C = s} d(s') \ = \sum_{s' \in \mathcal{E}(X)} \delta_{s'|C}(s) \cdot d(s').$$

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*Any factorizable (i.e. local) hidden-variable model defines a global section.*

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So:

existence of a local hidden-variable model for a given empirical model
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Hence:

No such h.v. model exists (the empirical model is **non-local**/**contextual**)
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# Methods for showing obstructions to global sections

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Thus in terms of well-known examples, we have

$$\text{Bell} < \text{Hardy} < \text{GHZ}$$

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If we measure each particle with a choice of $X$ or $Y$ observable, the probability for each outcome is given by the inner product

$$|\langle \mathrm{GHZ} | b_1 \cdots b_n \rangle \rangle|^2.$$

# GHZ States

In each finite dimension $n > 2$ we have the GHZ state, written in the $Z$ basis as

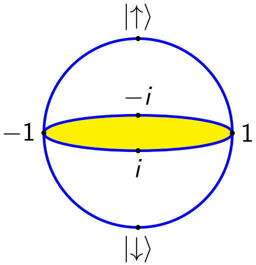
$$\frac{|\uparrow \cdots \uparrow\rangle \; + \; |\downarrow \cdots \downarrow\rangle}{\sqrt{2}}.$$

Physically, this corresponds to $n$ particles prepared in a certain entangled state.

If we measure each particle with a choice of $X$ or $Y$ observable, the probability for each outcome is given by the inner product

$$|\langle \mathrm{GHZ} | b_1 \cdots b_n \rangle\rangle|^2.$$

This computation is controlled by the product of the $|\downarrow\rangle$-coefficients of the basis vectors: cyclic group generated by $i \; \cong \; \mathbb{Z}_4$.



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- If the number of $Y$ measurements is $4k + 2$, the number of 1's in the outcomes is odd.

NB: a model with these properties can be realized in quantum mechanics.

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Thus for any $Y^{(i)}$, $Y^{(j)}$ assigned the **same** value, if we substitute X's in those positions they must receive **different** values. Similarly, for any $Y^{(i)}$, $Y^{(j)}$ assigned different values, the corresponding $X^{(i)}$, $X^{(j)}$ must receive the same value.

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Suppose not all $Y^{(i)}$ are assigned the same value. Then for some i, j, k, $Y^{(i)}$ is assigned the same value as $Y^{(j)}$, and $Y^{(j)}$ is assigned a different value to $Y^{(k)}$. Thus $Y^{(i)}$ is also assigned a different value to $Y^{(k)}$. Then $X^{(i)}$ is assigned the same value as $X^{(k)}$, and $X^{(j)}$ is assigned the same value as $X^{(k)}$. By transitivity, $X^{(i)}$ is assigned the same value as $X^{(j)}$, yielding a contradiction.

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Suppose not all $Y^{(i)}$ are assigned the same value. Then for some i, j, k, $Y^{(i)}$ is assigned the same value as $Y^{(j)}$, and $Y^{(j)}$ is assigned a different value to $Y^{(k)}$. Thus $Y^{(i)}$ is also assigned a different value to $Y^{(k)}$. Then $X^{(i)}$ is assigned the same value as $X^{(k)}$, and $X^{(j)}$ is assigned the same value as $X^{(k)}$. By transitivity, $X^{(i)}$ is assigned the same value as $X^{(j)}$, yielding a contradiction.

The remaining cases are where all Y's receive the same value. Then any pair of X's must receive different values. But taking any 3 X's, this yields a contradiction, since there are only two values, so some pair must receive the same value.

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- We can similarly define logical non-locality for states; we say that a state is logically non-local if for some choice of local observables, the resulting empirical model is logically non-local; while the state is **not** strongly non-local.

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- We say that a state is strongly non-local if for **some** choice of local observables for each party, the resulting empirical model is strongly non-local.
- We can similarly define logical non-locality for states; we say that a state is logically non-local if for some choice of local observables, the resulting empirical model is logically non-local; while the state is **not** strongly non-local.
- Finally, a state is weakly non-local if it is non-local, but neither of the previous two cases apply.

# The Characterization Problem

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We believe that an answer to this problem will shed considerable light on the structure of multipartite states, not least because it will necessitate solving the following task:

> *Given a multipartite state, find local observables which witness its highest degree of non-locality.*

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For example, the only strongly contextual bipartite models are the PR-boxes, which are of course not quantum realizable. By contrast, for all $n > 2$, the $n$-partite GHZ states are strongly contextual.

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*All bipartite entangled states* **except** *the maximally entangled ones are logically non-local.*

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However, as we shall see, for $n > 2$ a different picture emerges.

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Let $P(n)$ be the class of $n$-qubit pure states which, up to permutation, can be written as tensor products of 1-qubit and 2-qubit maximally entangled states. Let $L(n)$ be the set of logically non-local $n$-qubit states.

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The general result is proved non-constructively.

# Permutation Symmetric States

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For each $n \geq 2$, $0 < k < n$ the Dicke state $S(n, k)$ is defined as:

$$S(n,k) \ := \ K \sum_{\text{perm}} |0^k 1^{n-k}\rangle.$$

where $K = \binom{n}{k}^{-1/2}$ is a normalization constant, and we sum over all products of $k$ 0-kets and $n - k$ 1-kets.

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*All permutation symmetric states are logically non-local.*

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$$\Psi_F = \frac{1}{\sqrt{2^n}} \sum_{q_1 q_2 \ldots q_n = 00\ldots 0}^{11\ldots 1} |q_1 q_2 \ldots q_n F(q1, q2, \ldots, q_n)\rangle$$

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*All balanced functionally dependent states are in $P(n)$ or $L(n)$.*