Contextual Semantics: From Quantum Mechanics to Logic, Databases, Constraints, and Complexity Lecture 3

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Recap of Basic Mathematical Setting

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A setting for **contextual probability**.

Empirical Models, Global Sections and Contextuality

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The model is **contextual** if there is no such global section.

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Incidence matrix for $(2, 2, 2)$ is 16×16.

The $(2, 2, 2)$ Incidence Matrix

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$$
\begin{bmatrix}
1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\
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This matrix has rank 9.

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In general, the matrix for $(n, 2, 2)$ has rank 3^n. This is a special case of a much more general result we will describe later.

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Hence solutions correspond exactly to global sections — which as we have seen, correspond exactly to local hidden-variable realizations!

The Bell Model

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	$(0,0)$	$(1,0)$	$(0,1)$	$(1,1)$
(a,b)	$1/2$	0	0	$1/2$
(a',b)	$3/8$	$1/8$	$1/8$	$3/8$
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Solutions in the non-negative reals: this corresponds to solving the linear system over $\mathbb{R}$, subject to the constraint that $\mathbf{X} \geq \mathbf{0}$ (linear programming problem).

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Proof We focus on 4 out of the 16 equations, corresponding to rows 1, 6, 11 and 13 of the incidence matrix. We write X_i rather than $\mathbf{X}[i]$.

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Since all these numbers must be non-negative, the left-hand side of this equation must be greater than or equal to the left-hand side of the first equation, yielding the required contradiction. □

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Proposition

The following are equivalent:

1. $\mathbf{1} \cdot \mathbf{X}^* = 0$.
2. *The model is strongly contextual.*

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Experimental tests for contextuality ...

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Now we are interested in solutions over the **boolean semiring**, *i.e.* a boolean satisfiability problem. E.g. the equation specified by the first row of the incidence matrix gives the clause

$$X_1 \ \vee \ X_2 \ \vee \ X_3 \ \vee \ X_4$$

while the fifth yields the formula

$$\neg X_1 \ \wedge \ \neg X_3 \ \wedge \ \neg X_5 \ \wedge \ \neg X_7.$$

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Proposition

The possibilistic Hardy model has no global section over the booleans.

The 'Hardy paradox'

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Since every disjunct in the first formula appears as a negated conjunct in one of the other three formulas, there is no satisfying assignment. □

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Conclusion: Bell $<$ Hardy.

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Thus negative probabilities characterize the no-signalling rather than the quantum realm.

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For 18-vector K-S construction, $D = 118$; for Peres-Mermin square, $D = 34$.

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We say that $\mathcal{M}$ is **homogeneous** if the following conditions hold:

1. All the contexts $C \in \mathcal{M}$ have the same number n of elements.
2. Every set $U \in \mathcal{U}^{(j)}$ is a subset of the same number N_j of contexts $C \in \mathcal{M}$. Note that we must always have $N_0 = p$, where $p = |\mathcal{M}|$.

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In this case, $p = 6$, $n = 3$, $N_1 = 2$, and $N_2 = N_3 = 1$.

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- The corresponding value for the Peres-Mermin square is $D = 34$, with ambient dimension 48.

Rank of the Incidence matrix

We can also apply this formula to the rank of the incidence matrix. For example, for $(n, 2, 2)$-scenarios, where the incidence matrix is of size $4^n \times 4^n$, the rank is 3^n. This formula for the rank can be made visually apparent in this case, by noting that, with a suitable choice of enumeration for the rows and columns, the incidence matrices $\mathbf{M}(n)$ have a self-similar inductive structure:

$$\mathbf{M}(1) : \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} \qquad \mathbf{M}(n+1) : \begin{bmatrix} \mathbf{M}(n) & \mathbf{M}(n) & 0 & 0 \\ 0 & 0 & \mathbf{M}(n) & \mathbf{M}(n) \\ \mathbf{M}(n) & 0 & \mathbf{M}(n) & 0 \\ 0 & \mathbf{M}(n) & 0 & \mathbf{M}(n) \end{bmatrix}$$

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This vector can be taken as giving a **local hidden-variable realization of the PR box using negative probabilities**. Similar realizations can be given for the other PR boxes.

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There is a semiring homomorphism $\mathbb{R}_{\geq 0} \longrightarrow \mathbb{B}$ which induces a natural transformation

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The effect of applying this to a probabilistic model is exactly to produce the boolean model corresponding to its support: for each context C, the probability distribution $e_C \in \mathcal{D}_{\mathbb{R}_{\geq 0}}(O^C)$ is mapped to the finite non-empty subset

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This means that no-go theorems proved at the possibilistic level are stronger (in fact, **strictly** stronger) than those proved at the probabilistic level.

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As we shall now see, they arise very directly in a number of familiar CS settings.

Relational databases

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branch-name	**account-no**	**customer-name**	**balance**
Cambridge	10991-06284	Newton	£2,567.53
Hanover	10992-35671	Leibniz	€11,245.75
...	...	...	...

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Does this look familiar?

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Note that if $A \subseteq B$, then restriction is just **projection**. For $R \in \mathcal{D}_{\mathrm{B}}\mathcal{E}(B)$

$$R|_A := \{t|_A \,:\, t \in R\}.$$

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Note that if $A \subseteq B$, then restriction is just **projection**. For $R \in \mathcal{D}_{\mathrm{B}}\mathcal{E}(B)$

$$R|_A := \{t|_A \, : \, t \in R\}.$$

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The compatibility condition for an instance is **projection consistency**:

$$R_A|_{A \cap B} = R_B|_{A \cap B}$$

means that the two relations have the same projections onto their common set of attributes.

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Proposition

Let $(R_1, \ldots, R_k)$ be an instance for the schema $\Sigma = \{A_1, \ldots, A_k\}$. Define $R := \bowtie_{i=1}^{k} R_i$. Then a universal relation for the instance exists if and only if $R|_{A_i} = R_i$, $i = 1, \ldots, k$, and in this case R is the largest relation in $\mathcal{R}(\bigcup_i A_i)$ satisfying the condition for a global section.

A dictionary

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Relational databases	measurement scenarios
attribute	measurement
set of attributes defining a relation table	compatible set of measurements
database schema	measurement cover
tuple	local section (joint outcome)
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We can also consider probabilistic databases and other generalisations;
cf. provenance semirings.

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These formulas define possibilistic models; and their satisfying assignments

$$v : \mathcal{X} \longrightarrow O$$

correspond exactly to **global sections** of these models.

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Proposition

For each n, $\mathsf{HV}(n)$ *is in* NP*.*

Proof

From the previous Proposition, it is clear that $\mathsf{HV}(n)$ is defined by the following second-order formula interpreted over finite structures (U, e):

$$\forall \vec{x}. \exists \vec{y}. R(\vec{x}, \vec{y}) \;\wedge\; \forall \vec{x}, \vec{y}.\, R(\vec{x}, \vec{y}) \;\rightarrow\; \exists f_1, \ldots, f_n.\, \bigwedge_i f_i(x_i) = y_i \;\wedge\; \forall \vec{v}. R(\vec{v}, f(\vec{v})).$$

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By standard quantifier manipulations, this can be brought into an equivalent Σ^1_1 form, and hence HV(n) is in NP. □

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- These are used to show that HV(n), $n > 2$, is NP-complete; smaller instances are in PTIME.
- The robust paradigm is an interesting and non-trivial extension of current theory, and worthy of further study.

Contextual Semantics

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The **Contextual semantics hypothesis**: we can find common mathematical structure in all these diverse manifestations, and develop a widely applicable theory.

Valuation Algebras

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Subsequently understood to be applicable to a very wide range of inference problems. E.g. *Generic Inference* by Marc Pouly and Jürg Kohlas, Wiley 2011;

> *The inference task can then be described for general valuation algebras, which leads to a single computational problem that abstracts numerous important and seemingly different applications in computer science as for example query answering in databases, the evaluation of Bayesian and Gaussian networks, the solution of constraint, equation and inequality systems, satisfiability and theorem proving in logics, smoothing and filtering in linear dynamic systems and hidden Markov chains, the computation of discrete Fourier and Cosine transforms, various applications of path problems and coding schemes, sparse matrix techniques or numerical and symbolic partial differentiation. The properties of valuation algebras enable the efficient solution of all these problems with a single generic algorithm that exploits so-called tree-decomposition techniques.*

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Operations:

- Domain $d : \Phi \to D$; $\phi \mapsto d(\phi)$.
- Projection $\Phi \times D \to \Phi$; $(\phi, x) \mapsto \phi^{\downarrow x}$, for $x \subseteq d(\phi)$.
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Axioms:

- (A1) $(\Phi, \otimes)$ forms a commutative semigroup.
- (A2) $d(\phi \otimes \psi) = d(\phi) \cup d(\psi)$.
- (A3) $d(\phi^{\downarrow x}) = x$.
- (A4) $(\phi^{\downarrow y})^{\downarrow x} = \phi^{\downarrow x}, \quad (x \subseteq y \subseteq d(\phi))$.
- (A5) $(\phi \otimes \psi)^{\downarrow z} = \phi \otimes \psi^{\downarrow z \cap d(\psi)}, \quad (d(\phi) \subseteq z \subseteq d(\phi) \cup d(\psi))$.
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In most applications, valuations with domain x are distributions on assignments $\Pi_{X \in x} V_X$ valued in a semiring R. Projection is marginalisation.

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This leads to the fusion rule: if $x = \{X_1, \ldots, X_n\}$, then

$$(\phi_1 \otimes \cdots \otimes \phi_n)^{\downarrow x} \;=\; \bigotimes \mathsf{Fus}_{X_n}(\cdots(\mathsf{Fus}_{X_1}(\{\phi_1, \ldots, \phi_n\}))\cdots)$$

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The Shafer-Shenoy architecture is the most basic, and works for any valuation algebra. The L-S and HUGIN architectures achieve more efficiency, assuming additional properties of the valuation algebra (divisibility), while logical inference can be performed if we assume idempotence.

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$$\otimes_{x,y} : \mathcal{F}(x) \times \mathcal{F}(y) \to \mathcal{F}(x \cup y).$$

  The properties of this map given by (A1) and (A5) say that it is a natural transformation making $\mathcal{F}$ into a **symmetric monoidal functor**, with respect to the monoidal structure on D as a join semilattice, and $(\mathbf{Set}, \times)$ as a monoidal category.

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- In particular, the crucial axiom (A5) is (essentially) **naturality**.

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This may suggest a novel extension of DL with combination "built in", which may be well adapted to studying generic inference as captured by valuation algebras.

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However, using the cover

$$f_1 : x \mapsto a, \quad f_2 : y \mapsto b, \quad f_3 : z \mapsto b$$

we do have a gluing:

$$s = \{John(a), Man(a), donkey(b), \neg Man(b), grey(b)\}.$$

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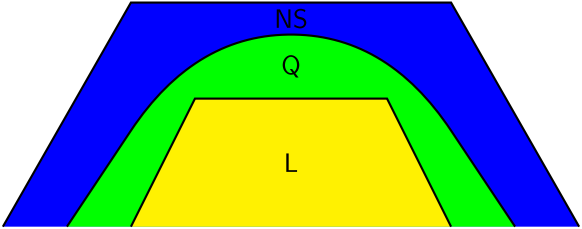

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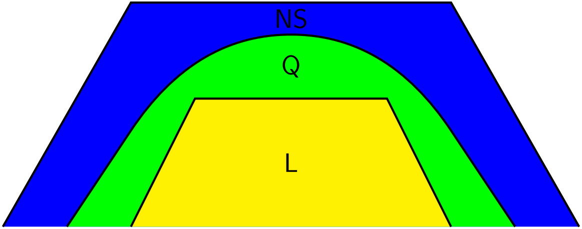

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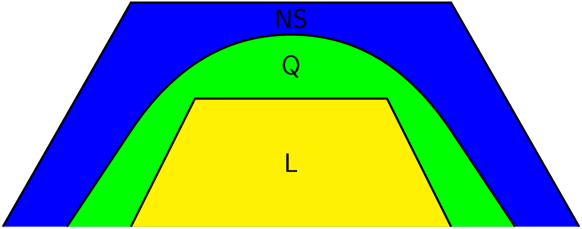

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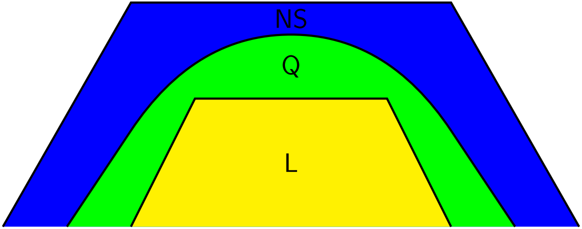

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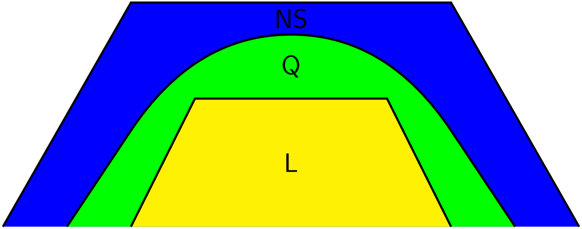

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- Quantum measures on global assignments give quantum models?

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- For each $i \in \mathbf{n}$, $m \in M_i$, and $o \in O_i$, a unit vector $\psi_{m,o}$ in $\mathcal{H}_i$, subject to the condition that the vectors $\{\psi_{m,o} \, : \, o \in O_i\}$ form an orthonormal basis of $\mathcal{H}_i$.

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We define a relational empirical model $e \subseteq M \times O$ by

$$e(\overline{m}, \overline{o}) \;\equiv\; p_{\overline{m}}(\overline{o}) > 0.$$

Thus e arises as the 'possibilistic collapse' of the usual quantum formalism.

Quantum Realization of the Hardy Model

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We consider the two-qubit system, with X_2 and Y_2 measurement in the computational basis. We take $R = 0$, $G = 1$. The eigenvectors for X_1 are taken to be

$$\sqrt{\frac{3}{5}}|0\rangle + \sqrt{\frac{2}{5}}|1\rangle, \qquad -\sqrt{\frac{2}{5}}|0\rangle + \sqrt{\frac{3}{5}}|1\rangle$$

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The possibilistic collapse of this model is thus a Hardy model.

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The class QM(d) *is in* PSPACE*. That is, there is a* PSPACE *algorithm to decide, given an empirical model, if it arises from a quantum system of dimension* d*.*

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This fragment has PSPACE complexity (Canny). Moreover, the sentence can be constructed in polynomial time from the given relational empirical model. Hence membership of QM is in PSPACE.

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Seems hard . . .

References

Papers (available on arXiv):

- S. Abramsky and A. Brandenburger. The sheaf-theoretic structure of non-locality and contextuality. *New Journal of Physics*, 13(2011):113036, 2011.
- S. Abramsky, S. Mansfield and R. Soares Barbosa, The Cohomology of Non-Locality and Contextuality, in *Proceedings of QPL 2011*, EPTCS 2011.
- S. Abramsky and L. Hardy. Logical Bell Inequalities. *Phys. Rev. A* 85, 062114 (2012).
- S. Abramsky, Relational Hidden Variables and Non-Locality. *Studia Logica* 101(2), 411–452, 2013.
- S. Abramsky, G. Gottlob and P. Kolaitis, Robust Constraint Satisfaction and Local Hidden Variables in Quantum Mechanics, Proceedings IJCAI 2013.
- S. Abramsky, Relational Databases and Bell's Theorem, In *In Search of Elegance in the Theory and Practice of Computation: Essays Dedicated to Peter Buneman*, Springer 2013.
- S. Abramsky and M. Sadrzadeh, Semantic Unification: a sheaf theoretic approach to natural language, in *Categories and Types in Logic, Language and Physics: Essays Dedicated to Jim Lambek*, Eds. Claudia Casadio, Bob Coecke, Michael Moortgat, Philip Scott, Springer 2014.