Review of Background Material on Discrete Mathematics and Probability

Samson Abramsky

Set theory notations

We shall assume basic notations of elementary set theory:

set membership	$x \in S$
set inclusion	$S \subseteq T$
union	$S \cup T$, $\bigcup_{i \in I} S_i$
intersection	$S \cap T$, $\bigcap_{i \in I} S_i$
powerset	$\mathscr{P}(S)$
relative complement	S^c
set difference	$S \setminus T$

Ordered Pairs and Relations

We take as fundamental the notion of **ordered pair** (x, y). (This can be defined in set theory, but we shall not need this.)

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The cartesian product of sets X, Y is defined as

$$X \times Y \;:=\; \{(x, y) \mid x \in X \;\wedge\; y \in Y\}.$$

A **relation** from a set X to a set Y is a subset $R \subseteq X \times Y$. We shall use the notation $R : X \multimap Y$ to indicate that R is a relation from X to Y.

Composition of Relations

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Given relations $R : X \multimap Y$ and $S : Y \multimap Z$, we can compose them: $R; S : X \multimap Z$ is defined by

$$R; S \;:=\; \{(x, z) \mid \exists y.\, (x, y) \in R \;\wedge\; (y, z) \in S\}.$$

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Proposition

Composition of relations is associative, and has the diagonal relations as identities. Given $R : X \multimap Y$, $S : Y \multimap Z$, $T : Z \multimap W$:

$$(R; S); T \;=\; R; (S; T)$$

and

$$\mathrm{id}_X; R \;=\; R \;=\; R; \mathrm{id}_Y$$

where $\mathrm{id}_X := \{(x, x) \mid x \in X\}$.

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Proposition

We have $R^{\cup\cup} = R$, *and* $(R; S)^{\cup} = S^{\cup}; R^{\cup}$.

Functions

Functions

A **function** f from X to Y is a relation from X to Y with the following additional properties:

Single-valuedness: $(x, y) \in f$ and $(x, z) \in f$ implies that $y = z$.

Totality: for all $x \in X$, for some $y \in Y$, $(x, y) \in f$.

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Given functions $f : X \to Y$ and $g : Y \to Z$, we can compose them:

$$g \circ f : X \to Z, \qquad g \circ f(x) := g(f(x)).$$

This agrees with the composition of g and f as relations.

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It is **bijective** if there is a function $g : Y \to X$ such that $g \circ f = \mathrm{id}_X$ and $f \circ g = \mathrm{id}_Y$. We write $X \cong Y$ if there is a bijection $f : X \to Y$.

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Exercises

1. Show that a function is bijective if and only if it is injective and surjective.
2. Show that $\mathscr{P}(X) \cong \mathbf{2}^X$, where $\mathbf{2} := \{0, 1\}$.
3. Show that a relation $R : X \multimap Y$ is single-valued if and only if $R^\cup ; R \subseteq \mathrm{Id}_Y$.
4. Show that R is total if and only if $\mathrm{Id}_X \subseteq R ; R^\cup$.
5. Show that a function $f : X \to Y$ is injective if and only if $f ; f^\cup \subseteq \mathrm{Id}_X$.
6. Show that a function $f : X \to Y$ is surjective if and only if $\mathrm{Id}_Y \subseteq f^\cup ; f$.

Disjoint families, restriction

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Given a function $f : X \to Y$, there is a function $f^{-1} : \mathscr{P}(Y) \to \mathscr{P}(X)$, the **inverse image** of f, defined by:

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A family of sets $\{X_i\}_{i \in I}$ is **pairwise disjoint** if $X_i \cap X_j = \varnothing$ whenever $i \neq j$. The union of such a pairwise disjoint family is called a **disjoint union**: given any $x \in \bigcup_{i \in I} X_i$, there is a unique i such that $x \in X_i$.

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Given a function $f : X \to Y$ and $U \subseteq X$, the **restriction** of f to U, written $f|_U : U \to Y$, is defined by

$$f|_U(x) \;:=\; f(x), \quad x \in U.$$

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Exercise Suppose we have functions $f : U \to Y$ and $g : V \to Y$. Give a necessary and sufficient condition for there to exist a function $h : U \cup V \to Y$ such that $h|_U = f$ and $h|_V = g$.

Probability distributions and measures on finite sets

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This function satisfies the following properties:

$$(P1) \quad p(X) \;=\; 1$$

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We call a function $p : \mathscr{P}(X) \to [0, 1]$ satisfying (P1) and (P2) a **probability measure**.

Equivalence of distributions and measures

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By induction, $(P2)$ extends to any finite disjoint union: if $A_1, \ldots, A_n$ are pairwise disjoint, then

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Proposition

Each probability distribution $d : X \to [0,1]$ determines a probability measure $p : \mathscr{P}(X) \to [0,1]$ defined by

$$p(A) \;:=\; \sum_{x \in A} d(x).$$

Conversely, each probability measure $p : \mathscr{P}(X) \to [0,1]$ determines a probability distribution $d : X \to [0,1]$, defined by

$$d(x) \;=\; p(\{x\}).$$

These maps between probability distributions and measures are mutually inverse.

Additional properties of probability measures

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Proposition

A probability measure satisfies the following additional properties:

$$(P3) \quad p(A^c) = 1 - p(A)$$

$$(P4) \quad p(\varnothing) = 0$$

$$(P5) \quad p(A \cup B) = p(A) + p(B) - p(A \cap B)$$

$$(P6) \quad A \subseteq B \Rightarrow p(A) \leq p(B)$$

$$(P7) \quad p(\bigcup_{i=1}^{n} A_i) \leq \sum_{i=1}^{n} p(A_i)$$

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&(P6) \quad A \subseteq B \;\Rightarrow\; p(A) \leq p(B) \\
&(P7) \quad p\left(\textstyle\bigcup_{i=1}^{n} A_i\right) \;\leq\; \sum_{i=1}^{n} p(A_i)
\end{aligned}
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Proposition

Let $B_1, \ldots, B_n$ be a pairwise disjoint family of sets, with $\bigcup_{i=1}^{n} B_i = X$. Then, for any set A:

$$p(A) \;=\; \sum_{i=1}^{n} p(A \cap B_i).$$

Conditional Probability

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Proposition (Law of Total Probability)

Let $B_1, \ldots, B_n$ be a pairwise disjoint family of sets, with $\bigcup_{i=1}^{n} B_i = X$. Then, for any set A:

$$p(A) = \sum_{i=1}^{n} p(A|B_i)p(B_i).$$

Bayes' Theorem

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We now state Bayes' Theorem in its most commonly encountered form.

Proposition (Bayes' Theorem)

Let $B_1, \ldots, B_n$ be a pairwise disjoint family of sets, with $\bigcup_{i=1}^{n} B_i = X$. Then, for any set A:

$$p(B_k|A) = \frac{p(A|B_k)p(B_k)}{\sum_{i=1}^{n} p(A|B_i)p(B_i)}.$$

Example of use of Bayes' Theorem

An HIV test gives a positive result with probability 98% when the patient is indeed affected by HIV, while it gives a negative result with 99% probability when the patient is not affected by HIV. If a patient is drawn at random from a population in which 0.1% of individuals are affected by HIV and he is found positive, what is the probability that he is indeed affected by HIV?

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The given information can be summarised as follows:

$$
\begin{aligned}
p(\text{positive}|\text{HIV}) &= 0.98 \\
p(\text{positive}|\text{NO HIV}) &= 1 - 0.99 = 0.01 \\
p(\text{HIV}) &= 0.001 \\
p(\text{NO HIV}) &= 0.999
\end{aligned}
$$

Example ctd

The probability of being found positive can be derived using the law of total probability:

$$\begin{aligned} p(\text{positive}) &= p(\text{positive}|\text{HIV})p(\text{HIV}) + p(\text{positive}|\text{NO HIV})p(\text{NO HIV}) \\ &= 0.98 \cdot 0.001 + 0.01 \cdot 0.999 \\ &= 0.00098 + 0.00999 \\ &= 0.01097 \end{aligned}$$

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So, by Bayes' Theorem:

$$\begin{aligned} p(\text{HIV}|\text{positive}) &= \frac{p(\text{positive}|\text{HIV})p(\text{HIV})}{p(\text{positive}} \\ &= \frac{0.98 \cdot 0.001}{0.01097} \\ &= \frac{0.00098}{0.01097} \simeq 0.08933. \end{aligned}$$

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So despite the conditional accuracy of the test, the unconditional probability of being affected by HIV when found positive is less than 10%.

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Let A and B be events with $p(A) > 0$ and $p(B) > 0$. The following conditions are equivalent:

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If any of these equivalent conditions are satisfied, we say that A and B are **independent**. We write $A \perp\!\!\!\perp B$.

Conditional Independence

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Let A, B and C be events with $p(A \cap C) > 0$ and $p(B \cap C) > 0$. The following conditions are equivalent:

$$(1) \quad p(A|B,C) = p(A|C)$$

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If any of these equivalent conditions are satisfied, we say that A and B are **conditionally independent given** C. We write $A \perp\!\!\!\perp B \mid C$.

Products and Marginals

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We consider a probability distribution $d : X \times Y \to [0, 1]$ on the cartesian product of finite sets X, Y. This induces a probability distribution $d_X : X \to [0, 1]$, defined by:

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This is the **marginal** of d on X. Similarly, there is a marginal d_Y on Y.

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If we have two random variables $R : X \to V$, $S : X \to W$ defined on X, we have a **joint distribution**:

$$p(R = v, S = w) \ := \ p(\{x \in X \mid R(x) = v \ \wedge \ S(x) = w\}).$$

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Given $f : X \to Y$, we define a function $\mathrm{Prob}(f) : \mathrm{Prob}(X) \to \mathrm{Prob}(Y)$, which maps probability distributions on X to probability distributions on Y, by "pushing them forwards along f". This is defined by $\mathrm{Prob}(f)(d) = d'$, where $d' \in \mathrm{Prob}(Y)$ is defined by:

$$d'(y) \;=\; \sum_{f(x)=y} d(x) \;=\; p(f^{-1}(y)),$$

where p is the probability measure $p : \mathscr{P}(X) \to [0, 1]$ corresponding to the distribution d.

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Proposition

If $f : X \to Y$ *and* $g : Y \to Z$, *then* $\mathrm{Prob}(g \circ f) = \mathrm{Prob}(g) \circ \mathrm{Prob}(f)$. *Also,* $\mathrm{Prob}(\mathrm{id}_X) = \mathrm{id}_{\mathrm{Prob}(X)}$.

Complex numbers

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Remarkably, **all** algebraic equations over $\mathbb{C}$ have solutions in $\mathbb{C}$. (The "Fundamental Theorem of Algebra", or in modern terminology, $\mathbb{C}$ is algebraically closed.)

The Complex Plane

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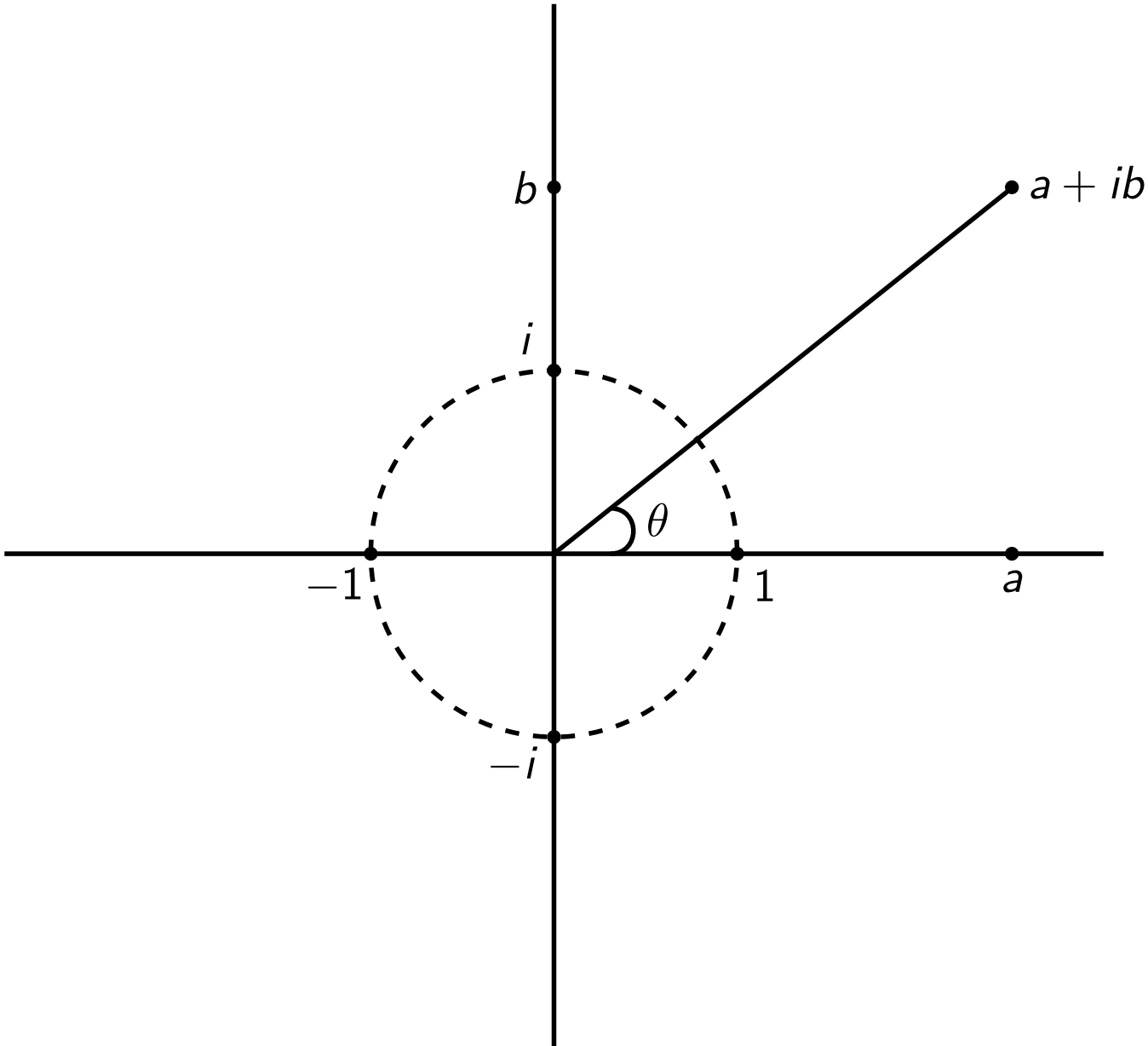
Complex numbers can be viewed geometrically, as 2-dimensional vectors.



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Also,

$$z\bar{z} \;=\; (a + ib)(a - ib) \;=\; a^2 + b^2.$$

So $z\bar{z} = |z|^2$.

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So

$$\begin{aligned} |e^{i\theta}|^2 &= e^{i\theta}\overline{e^{i\theta}} \\ &= (\cos\theta + i\sin\theta)(\cos\theta - i\sin\theta) \\ &= \cos^2\theta + \sin^2\theta \;=\; 1. \end{aligned}$$