The many faces of Proof Theory

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Abstract

Proof theory can be considered as the meta-theory of how mathematicians and logicians do their work. The questions therein turn around as simple things as *What is a proof?*, *What methods do we use to prove things?*. While mathematicians in general prefer to take a very liberal approach in what methods they use to prove theorems, logicians and in particular those from proof theory seek to reduce external influence and deduce from the very structure of the proof information about the problem and theory involved.

This week should serve as a general introduction to proof theory and its use in several different areas, hoping that its value and manyfold usage scenarios might inspire future proof theory logicians.

At the same time we present some of our main working areas, partly discussing current research topics and problems.

Introduction to Proof Theory (NP)

We start with a gentle introduction to Gentzen style proof theory by presenting Gentzen's sequent calculus for classical and intuitionistic logic. Many examples will support getting acquaintained with this way of proving. Continuing the fundamental cut-elimination result for **LK** will be discussed and the core ideas of the proof presented. The consequences of the cut-elimination theorem, e.g., the mid-sequent theorem, will give a good starting point into a short excursion to a completely different topic, namely using proof theory in projective geometry and formalizing the usage of sketches therein. Time allowed we will mimic Orevkov's speed-up result to show that also in projective geometry using cuts can be non-elementary faster than sketches.

Substructural Logics I & II (HO)

Study of substructural logics can be regarded as an enterprise of understanding various nonclassical logics in an uniform framework. Around the middle of the 1980s, some people independently discussed logics which are formalized in Gentzen-type sequent systems, like commutative version of Lambek calculus for categorial grammar, linear logic and logics lacking the weakening rule. A common feature of them is that these sequent systems lack some *structural rules*, which standard sequent system **LK** for classical logic or **LJ** intuitionistic one have. Gradually it has been discovered that many of nonclassical logics fall under this class. For instance, relevant logics do not allow the axiom $\alpha \to (\beta \to \alpha)$, which corresponds to the (left) *weakening rule* in sequent systems, and Łukasiewicz's many-valued logics do not allow the axiom $(\alpha \to (\alpha \to \beta)) \to (\alpha \to \beta)$, which corresponds to the *contraction rule*.

After discussions on structural rules, we will introduce the basic sequent system **FL** for substructural logics and show that many nonclassical logics can be regarded as extensions of **FL**. Then, we will discuss the following properties for substructural logics from a proof-theoretic point of view, as long as time allows: Disjunction property, Craig interpolation property (Maehara's method), Local deduction theorem, Deductive interpolation property, Decidability of logics without contraction rule, Decidability of **FLec** (Kripke's argument).

Gödel logics (NP)

Gödel logics are one of the oldest families of many-valued logics. Propositional finite-valued Gödel logics were introduced by Gödel in 1933 to show that intuitionistic logic does not have a characteristic finite matrix. They provide the first examples of intermediate logics. Dummett was the first to study infinite valued propositional Gödel logics, axiomatizing the set of tautologies over infinite truth-value sets by intuitionistic logic extended by the linearity axiom $(A \to B) \vee (B \to A)$. Hence, infinite-valued propositional Gödel logic is sometimes called Gödel-Dummett logic or Dummett's LC. In terms of Kripke semantics, the characteristic linearity axiom picks out those accessibility relations which are linear orders.

We will start with introducing these logics, i.e., their syntax and semantics, and give an overview of the known properties and results as obtained in the last 20 years. Then we turn to proof theory and introduce an extension of Gentzen's sequent calculus called hypersequent calculus. The method of hypersequents for the axiomatization of non-classical logics was pioneered by Avron. Hypersequent calculi are especially suitable for logics that are characterized semantically by linearly ordered structures, among them Gödel logics.